

Non-Affine Stein Manifolds and Normal Crossing Divisors

Randall R. Van Why

October 27, 2026

Abstract

We show that there are Stein manifolds that admit compactifications by normal crossings divisors despite being neither affine nor quasi-projective. To achieve this, we study the contact boundaries of neighborhoods of symplectic normal crossing divisors via a contact-geometric analog of W. Neumann's plumbing calculus. In particular, we give conditions under which the neighborhood is determined by the contact structure on its boundary.

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Acknowledgements

This work was primarily supported by NSF Grant DMS-1502632. We thank the institutions of Northwestern University, Princeton University, and the Georgia Institute of Technology for the hospitality during the completion of this project. Much thanks to Austin Christian, John Etnyre, David Gay, Jie Min, Emmy Murphy, Hadrian Quan, and Eric Zaslow for the numerous helpful conversations and input.

1 Motivation and summary of results

We seek to study compactifications of a Stein surface X . By compactification, we mean a complex analytic 4-manifold M admitting a subvariety $D \subset M$ such that $M - D$ is biholomorphic to X . The consideration of these objects gives rise to some interesting motivating questions:

- When is a Stein manifold X an algebraic variety (affine or quasi-projective)?
- When does a Stein manifold admit an algebraic compactification?
- Which algebraic Stein manifolds admit non-algebraic compactifications?
- How are the compactifications of X related to each other?

In this article, we consider these questions purely in the context of symplectic geometry where the rigidity of symplectic structures can lend a hand. Every Stein manifold X admits a symplectic structure and the general symplectic-topological underpinnings of Stein manifolds were characterized by Eliashberg in [Eli90].

1.1 Compactifications and non-Kähler manifolds

If X is a complex affine or quasi-projective variety, we may always *projectivize* X to obtain a (possibly singular) projective variety $\tilde{M}$, its projective completion. By construction, the intersection of the projective completion with an affine chart in projective space is biholomorphic to our original affine variety X . The set $\tilde{D} = \tilde{M} - X$ is a subvariety of $\tilde{M}$ called a *compactifying divisor*.

It is a classical result of Hironaka [Hir64a, Hir64b] that the singularities of $\tilde{M}$ can be completely resolved so as to produce a smooth projective variety M . This resolution process is carried out by blowing up points lying on the compactifying divisor and can be performed in such a way that the resulting manifold M still contains a biholomorphic copy of X . The complement $D := M - X$ is a subvariety with particularly nice singularities: (simple) normal-crossing singularities. The main topological conclusion we can draw from this story is summarized by:

Fact: Every affine variety X admits a smooth (projective) compactification M by a normal crossing divisor.

A similar picture emerges with quasi-projective varieties which may be presented as the complements of normal crossing divisors in projective varieties. A symplectic geometer would recognize a Stein manifold as an example of a Liouville manifold and, through that perspective, may exploit any number of different symplectic invariants. In [McL12], via an approach first introduced by Seidel [Sei08], McLean established an obstruction to a Stein manifold X being symplectomorphic to an affine variety relying on the growth rate of symplectic homology (c.f. [Sei08]). Through use of this obstruction, McLean has been able to determine, for example, when a cotangent bundle may be symplectomorphic to an affine variety [McL12, Corollary 1.3]. These growth rate techniques take a view toward the above compactification phenomenon.

McLean and Seidel's techniques essentially determine whether this sort of compactification is even possible for a given Liouville manifold X . In particular, they obstruct X from being symplectomorphic to an affine variety by showing it cannot be compactified by a normal crossing divisor. One would naively hope that the existence of such a compactification is sufficient for X to be symplectomorphic to some affine (or at least quasi-projective) variety. Our results will demonstrate

that this is not the case.

Theorem 1.1. *There exists a Stein manifold admitting a normal crossing divisor compactification that is not biholomorphic to any affine variety.*

To describe this Stein manifold more thoroughly, we recall a classic construction in symplectic geometry. In [Thu76], Thurston presented the first example of a closed non-Kähler symplectic manifold. Before the explication of this example, it was not known whether there were any significant differences between Kähler and symplectic manifolds. Thurston's example is 4-dimensional and was first studied in a separate context by Kodaira [Kod64] who showed that it admits an integrable almost complex structure and is thus a complex manifold. Because of this, the example has been known in the literature as the *Kodaira-Thurston manifold* which we will denote by M_{KT} .

In Kodaira's original presentation, M_{KT} is taken to be $(Nil^3/H) \times S^1$ where Nil^3 is the Heisenberg group and $H \subset Nil^3$ is a lattice. We will adopt Thurston's presentation in [Thu76] as the product $\mathcal{M}_\phi^3 \times S^1$ where $\mathcal{M}_\phi^3$ is the *mapping torus* of a right handed Dehn twist $\phi: T^2 \rightarrow T^2$. The diffeomorphism type of $\mathcal{M}_\phi$ is independent of the curve about which we perform the Dehn twist. The manifold M_{KT} cannot be Kähler because we have $b_1(M_{KT}) = 3$ [Thu76]. Kähler manifolds possess a natural Hodge structure and their first Betti number is always even [GH78, Chapter 7].

In [Gom95], Gompf showed that, for even dimensions $2n \geq 4$, any finitely presented group can be made the fundamental group of some closed non-Kähler symplectic $2n$ -manifold. Gompf's construction presents an avenue for finding many non-Kähler examples using obstructions similar to Thurston's. With all of this in mind, it is clear that being Kähler imposes restrictions on the underlying symplectic geometry, at least on the level of fundamental groups and first homology.

1.2 An interesting non-algebraic Stein manifold

Using the results contained in this work, we will construct a non-algebraic Stein 4-manifold X_{KT} given as the complement of a normal crossing divisor in the Kodaira-Thurston example. This will prove our original theorem.

Theorem 1.1. *Let M_{KT} be the Kodaira-Thurston manifold (see [Thu76]). Let $D_{KT} \subset M_{KT}$ denote the normal crossing divisor given by the union of a generic section with a fiber of M_{KT} , thought of as an elliptic fibration. Then the Stein manifold $X_{KT} = M_{KT} - D_{KT}$ is not biholomorphic to any affine variety.*

If X_{KT} were indeed biholomorphic to an affine variety X , then X would admit a compact Liouville subdomain $\bar{X} \subset X$ unique up to Liouville isomorphism whose completion is symplectic deformation equivalent to X_{KT} (see [Sei08, Section 4b] or c.f. [GS24, Section 2.4]). We are able to eliminate the possibility of the existence of such an X by studying the space of all *symplectic* normal crossing (or SNC^+) divisor compactifications of the Liouville domain $\bar{X}$ and relating them. We do this by considering gluings $M = \bar{X} \cup_{\partial\bar{X}} N_D$ where (N_D, ω) is a regular symplectic neighborhood of a normal crossing divisor D . The Stein structure of X endows $\bar{X}$ with the structure of a Liouville domain, a compact exact symplectic manifold with boundary. This structure furnishes $\partial\bar{X}$ with a natural contact structure. Because of this, we can phrase this as the study of SNC^+ divisor compactifications of the Liouville domain $\bar{X}$ obtained via a gluing along its contact boundary. We will adopt this context for the rest of our discussion.

1.3 Symplectic and contact topological results

Each SNC^+ divisor D has an associated decorated graph Γ_D which encodes topological information about the divisor. We call Γ_D the *divisor graph* of D . The main question we are interested in is:

Question: How much about the isomorphism type of Γ_D can we learn from the contactomorphism type of $(\partial\bar{X}, \xi)$?

We can see easily that answering this question makes steps toward understanding all SNC^+ compactifications of $\bar{X}$. Any such compactification (M, D) has an implicit contactomorphism $\Psi: (\partial\bar{X}, \xi_D) \rightarrow (\partial N_D, \xi_D)$ which defines the gluing between $\bar{X}$ and N_D . If we were able to discern information about Γ_D purely from this gluing map, we may be able to restrict the class of SNC^+ divisors D that may compactify X . In other words, we would understand something about the structure of the collection of all compactifications of X .

If one neglects symplectic, contact, and algebraic structures entirely and focuses only on the topological aspects of the above discussion, this perspective is well understood. The 3-manifolds $Y_D := \partial N_D$, henceforth known as *divisor boundaries*, are examples of *3-manifold plumbings* which are classical objects in low dimensional topology. The graphs Γ_D are examples of *plumbing graphs* which give a combinatorial description of how to perform a 3-manifold plumbing and construct Y_D topologically. In our discussion, we shall use “divisor graphs” and “plumbing graphs” essentially interchangeably with a preference toward “divisor graph.” We will distinguish the two wherever necessary.

In [Neu81], it was shown that divisor graphs can be “reduced” to simpler di-

visor graphs via a well-defined procedure without changing the diffeomorphism type of its associated divisor boundary. This reduction procedure always terminates and we are left with a graph Γ_D^{Top} which we will call the *topological reduction* of Γ_D . Neumann proved that, for most 3-manifold plumbings, the diffeomorphism type of the plumbing characterizes the isomorphism type of Γ_D^{Top} . Rephrased in terms of divisor boundaries, this theorem reads:

Theorem 1.2 (c.f. [Neu81], Theorem 4.2). *Let Y_D and $Y_{\tilde{D}}$ divisor boundaries with divisor graphs Γ_D and $\Gamma_{\tilde{D}}$. Then Y_D and $Y_{\tilde{D}}$ are diffeomorphic if and only if $\Gamma_D^{Top} \approx \Gamma_{\tilde{D}}^{Top}$.*

Throughout, we shall refer to this result as “Neumann’s theorem.” While this result does answer the question topologically, it is less useful for our purposes since Neumann’s reduction procedure has no regard (and actually may destroy) the contact topology at hand. To deal with this, we develop a new reduction procedure which terminates and produces a (possibly distinct) graph Γ_D^{ξ} , the *contact reduction* of Γ_D . In the case when the divisor boundary associated to Γ_D is a *prime* 3-manifold (c.f. [Hat], Chapter 1), by comparing Γ_D^{ξ} and Γ_D^{Top} , we can arrive at some topological conclusions:

Theorem 1.3. *Let N_D be a concave divisor neighborhood (Section 2.2). Then if ∂N_D is a prime 3-manifold and $\Gamma_D^{\xi} \not\approx \Gamma_D^{Top}$, one or more of the following are true:*

- (i) *We may blow up N_D to a concave SNC^+ divisor neighborhood $N_{\tilde{D}}$ which contains a smooth component S diffeomorphic to a sphere with $S \cdot S = 0$,*
- (ii) *$(\partial N_D, \xi_D)$ is Seifert-fibered over $\mathbb{RP}^2$ via a fibration with at most one singular fiber, or*
- (iii) *$(\partial N_D, \xi_D)$ contains an embedded incompressible Klein bottle.*

This structural theorem is implied by our main results for divisor boundaries (which are implicit in this result if one restricts to statements only about ∂N_D). The details of this will be discussed in Section 4.

Definition 1. We say a divisor boundary (Y_D, ξ_D) is *obstructed* if $\Gamma_D^{\xi} \not\approx \Gamma_D^{Top}$ and *unobstructed* otherwise.

In the unobstructed case, the following result follows directly from [Neu81].

Theorem 1.4. *Let (Y_D, ξ_D) and $(Y_{\tilde{D}}, \xi_{\tilde{D}})$ be a pair of unobstructed divisor boundaries with divisor graphs Γ_D and $\Gamma_{\tilde{D}}$. Suppose that (Y_D, ξ_D) and $(Y_{\tilde{D}}, \xi_{\tilde{D}})$ are contactomorphic. Then $\Gamma_D^{\xi} \approx \Gamma_{\tilde{D}}^{\xi}$.*

Thus Theorem 1.3 and Theorem 1.4 give a complete topological characterization of the extent to which Neumann's theorem fails to hold in the contact category. The proof of Theorem 1.1 essentially follows from a corollary of Theorem 1.4. The unobstructed case allows for to draw a strong conclusion about divisor neighborhoods:

Theorem 1.5. *Let (N_D, ω) and $(N_{\tilde{D}}, \tilde{\omega})$ denote concave divisor neighborhoods (Section 2.2) and suppose that $(\partial N_D, \xi_D)$ and $(\partial N_{\tilde{D}}, \xi_{\tilde{D}})$ are their associated divisor boundaries. Suppose that both D and $\tilde{D}$ are unobstructed. Then if $(\partial N_D, \xi_D)$ is contactomorphic to $(\partial N_{\tilde{D}}, \xi_{\tilde{D}})$, the symplectic manifolds (N_D, ω) and $(N_{\tilde{D}}, \tilde{\omega})$ are diffeomorphic up to blow-ups and blow-downs.*

The majority of the work in verifying that X_{KT} is non-algebraic, that is proving Theorem 1.1, is reduced to verifying that the divisor boundary associated to (M_{KT}, D_{KT}) is unobstructed in the above sense. From there we can use Theorem 1.5 to understand all concave SNC^+ divisor compactifications of X_{KT} which will allow us to conclude Theorem 1.1. The details of this will be handled in the latter part of Section 4.

To draw conclusions about compactifications, there is a subtlety involving mapping classes of diffeomorphisms used to glue on a divisor neighborhood. This is managed in the following result.

Theorem 1.6. *Let D be a concave compactifying SNC^+ divisor for a 4-dimensional Liouville domain (X^4, ω) and let (M^4, ω) be a compactification of X by D with associated mapping class $[\Psi] \in \pi_0(\text{Cont}(\partial X))$ defining the capping. If D is unobstructed, any other SNC^+ divisor compactification $(\tilde{M}, \tilde{\omega})$ with the same mapping class $[\Psi]$ is diffeomorphic to (M, ω) up to blow-ups and blow-downs.*

The outline of the paper is as follows: We will start with some necessary preliminaries (Section 2) surrounding the symplectic and contact geometric theory of concave symplectic normal crossing (or SNC^+) divisors. We then turn to the classical theory of plumbings, which will make up the main theoretical context for proving our primary lemma on divisor graph reductions. We will outline a framework and process for normalizing divisor graphs while preserving the contact structure on the divisor boundary (Section 3). After showing that we can almost normalize any divisor graph, we will collect all the topological ramifications into the main results of the paper (Section 4). We will then focus our attention to the particular case (M_{KT}, D_{KT}) above. We will use Theorem 1.5 to draw conclusions about X_{KT} and D_{KT} that will allow us to conclude with Theorem 1.1. A short discussion on birational symplectic geometry will follow.

2 Preliminaries

This section collects short definitions and descriptions of the geometric objects relevant to the sections that follow. Everything will be phrased in low dimensions as we will only focus on 4-dimensional varieties and symplectic manifolds with 3-dimensional contact-type boundaries. As a result, the divisors we concern ourselves with are all symplectic surface configurations.

2.1 Positive symplectic normal crossing divisors

Let (M, ω) be a symplectic 4-manifold. An embedded closed symplectic surface $\Sigma \subset M$ is called a *symplectic divisor*.

Definition 2. A subspace $D \subset (M^4, \omega)$ is called a (simple) *symplectic divisor with positive normal crossings* (or simply an SNC^+ -divisor) if we may write

$$D = \bigcup_{i=1}^N D_i$$

where $\{D_i\}_{i=1}^N$ is a collection of embedded closed symplectic surfaces in M such that any pair (D_i, D_j) with $D_i \cap D_j \neq \emptyset$ intersects ω -orthogonally only in transverse double points (i.e. all intersections must be modeled after intersecting coordinate planes in $\mathbb{C}^2$). The manifolds D_i are called the *smooth components* of the divisor D .

To each SNC^+ divisor D , we may construct a finite graph Γ_D encoding the connectivity information of D as follows:

- The vertex set of Γ_D consists of one vertex v_i for every component D_i .
- Two distinct vertices v_i, v_j are connected by an edge e_{ij}^k for each $p_k \in D_i \cap D_j \neq \emptyset$.

It is important to note that, with this definition, there are no edges in Γ_D from a vertex to itself (i.e. Γ_D is free of loops). Thus SNC^+ divisors, as defined above, are topologically identical to simple normal crossing divisors in the complex-algebraic setting.

Remark 1. For notational convenience, we will re-label the components of D by the corresponding vertices of Γ_D . Any vertex $v \in \Gamma_D$ corresponds to a component $D_v \subset D$. We let $\mathcal{E}(v)$ denote the set of edges in Γ_D adjacent to v and we let $\mathcal{N}(v)$ be the set of vertices neighboring v .

Regular neighborhoods of symplectic divisors are modeled after plumbings of disk bundles over the components. We let N_D be such smooth plumbing. The boundary ∂N_D is called a *divisor boundary* and is denoted by $Y_D := \partial N_D$ where N_D is the neighborhood of D discussed above.

2.2 Symplectic concavity near the boundary of a divisor

In many cases, closed neighborhoods of SNC^+ divisors will admit natural contact structures on their boundaries. These contact structures are due to the symplectic geometry near the boundary of the neighborhood. We say that the symplectic manifold with boundary N_D is *concave* near ∂N_D if it admits a collar neighborhood symplectomorphic to a portion of the negative symplectization of a contact manifold. In particular, this implies the presence of a local Liouville vector field V which is inwardly transverse to ∂N_D and has the property that $L_V \omega = -\omega$. This means the flow contracts the symplectic area over time.

Definition 3. An SNC^+ divisor D is said to be *concave* if every regular neighborhood N_D admits a closed sub-neighborhood N_D such that $(N_D, \omega|_{N_D})$ is concave near ∂N_D .

The Liouville vector field induces a contact structure ξ_D on the boundary $Y_D := \partial N_D$. The contactomorphism type of (Y_D, ξ_D) is stable under perturbation of V .

There is a simple condition that determines when N_D is concave. Every SNC^+ divisor, concave or otherwise, has an associated *intersection form* $Q_D: \Gamma_D \times \Gamma_D \rightarrow \mathbb{Z}$ defined by

$$Q_D(v, w) = D_v \cdot D_w$$

From this definition, $Q_D(v, w) = d$ if Γ_D has $d \geq 0$ edges from v to w and $Q_D(v, v)$ is the self-intersection number (in M) of D_v . Placing an arbitrary order on the vertices $v_1, \dots, v_N$, we regard the intersection form as a linear map $Q_D: \mathbb{R}^N \rightarrow \mathbb{R}^N$ defined, as a matrix, by the entries

$$(Q_D)_{ij} = Q_D(v_i, v_j).$$

We also have a well defined *area vector* $\underline{a} \in \mathbb{R}^N$ with entries

$$a_{v_i} = \int_{D_{v_i}} \omega.$$

Definition 4. We say that D satisfies the *positive GS-criterion* if there exists a vector $\underline{b} \in \mathbb{R}^N$ with positive entries solving

$$Q_D \underline{b} = \underline{a}.$$

We then have:

Theorem 2.1 ([LM19], **Theorem 1.1**). *If D satisfies the positive GS-criterion, then D has a concave neighborhood N_D inside any regular neighborhood U via the GS-construction ([LM], Section 2.1). The contactomorphism type of the boundary of N_D is independent of all choices made during the GS-construction.*

Remark 2. “GS” refers to the authors of [GS09] wherein a separate condition involving negative solutions is shown to be sufficient to ensure the existence of a *convex* neighborhood.

Keeping track of the full symplectic topological data of concave neighborhoods requires decorations on the graph Γ_D in the form of:

- Functions $(g, k): \text{Vert}(\Gamma_D) \rightarrow \mathbb{N}_{\geq 0} \times \mathbb{Z}$ where $g(v)$ is the genus and $k(v)$ is the self intersection number of the divisor D_v and $\text{Vert}(\Gamma_D)$ is the set of vertices of Γ_D .
- The symplectic area vector $\underline{a}$.

For the remainder of our discussion, all divisors will assumed to be concave and so we will assume that their (decorated) graphs satisfy the positive GS-criterion.

2.3 Divisor Boundaries

The fact that closed regular neighborhoods of concave divisors admit contact type boundaries means that to every such divisor D , we may associate a contact 3-manifold (Y_D, ξ_D) . The contactomorphism type of (Y_D, ξ_D) is stable up to perturbation of the local Liouville vector field. Any such 3-manifold Y_D is called a *divisor boundary* and the contact manifold (Y_D, ξ_D) is called a *contact divisor boundary*. As previously mentioned, divisor boundaries are examples of 3-manifold plumbings. These objects are classical to topology and have received extensive attention and use.

The diffeomorphism type of Y_D is completely determined by the divisor D . This follows from classical results in smooth topology. We are interested in understanding to what extent the diffeomorphism type of Y_D determines the contactomorphism type of (Y_D, ξ_D) .

2.4 Topological Plumbing Calculus

The diffeomorphism type of a divisor boundary (or even a more general plumbing) does not determine the isomorphism type of the divisor graph Γ_D . Performing a symplectic blow-up on $(N_D, d\lambda)$ at some point $p \in D$ away from the singular points will produce a new symplectic 4-manifold $(N_{\tilde{D}}, \tilde{\omega})$ which is a regular neighborhood of a divisor $\tilde{D}$ which reduces to D after blowing-down the exceptional curve. Blowing up is a local construction and does not affect the diffeomorphism type of the boundary $\partial N_{\tilde{D}}$. We thus have:

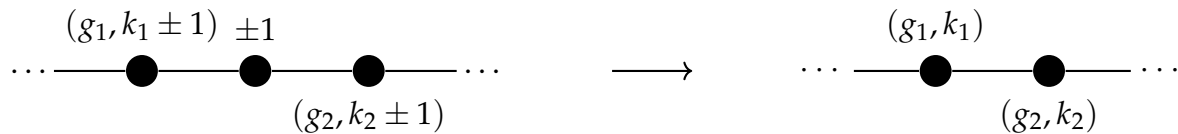
Proposition 1. *If Y_D and $Y_{\tilde{D}}$ are divisor boundaries, then we may have $Y_D \approx_{\text{Diff}} Y_{\tilde{D}}$ with $\Gamma_D \not\approx \Gamma_{\tilde{D}}$.*

We conclude that we cannot possibly recover Γ_D from the divisor boundary alone unless we somehow deal with ambiguities such as this. Fortunately, there is still a great deal of rigidity in the relationship between divisor boundaries and their associated graphs.

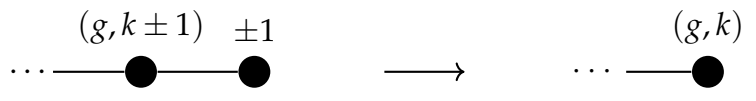
In [Neu81], the author defined a collection of eight moves which may be performed on a plumbing graph Γ (see [Neu81], Section 2). Each of these moves have associated topological constructions which transform any associated Y_Γ into a different plumbing $Y_{\tilde{\Gamma}}$ while preserving the diffeomorphism type.

There are only two moves which are relevant to this manuscript. They are named and defined diagrammatically by:

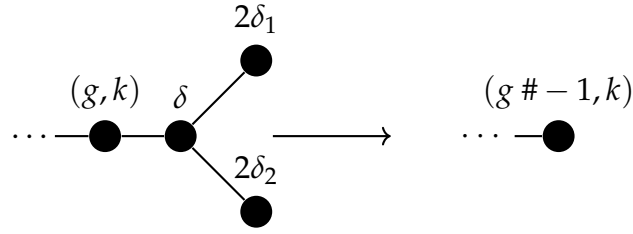
(1) The ± 1 blow-down: Defined by



or



and **(2) $\mathbb{RP}^2$ -absorption**: Defined by



Here $\delta_i = \pm 1$, $\delta = \frac{\delta_1 + \delta_2}{2}$, and the vertex with decoration $(g \# - 1, k)$ denotes the S^1 bundle over $\Sigma_{g \#} \mathbb{RP}^2$ with Euler number k .

The other moves are (by name only):

- (3) 0-curve absorption: This move obviously implies that the graph contains a sphere of self intersection number zero. These graphs will be discussed in the contact setting below.
- (4) Unoriented handle absorption: This move is not applicable to SNC^+ divisors as it involves reducing to a graph with a non-orientable vertex.
- (5) Oriented handle absorption: This move is equivalent to having a sphere of self-intersection number zero. This situation is dealt with in the contact setting below.
- (6) Splitting: All of our divisor graphs will be connected and do not admit any splittings.
- (7) Seifert graph exchanges: This performs a number of graph exchanges between components with one vertex and one loop and the standard star-shaped presentations of the Seifert-fibered spaces they represent. Because none of our graphs contain any loops (i.e. edges from a vertex to itself), these exchanges will never be applicable.
- (8) Annulus absorption: This move is performed on general plumbings with boundary. Since none of our divisor boundaries have boundaries of their own, this move does not apply.

Definition 5. Two plumbing graphs Γ_1, Γ_2 are said to be *TPC-combinatorially related*¹ if we can obtain one from the other via a sequence of moves in Neumann's plumbing calculus (see [Neu81], Section 2).

¹TPC = Topological Plumbing Calculus

Which brings us to one of the main results of Neumann's paper:

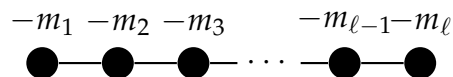
Theorem 2.2 ([Neu81], **Theorem 3.1**). *If Y_Γ is diffeomorphic to $Y_{\tilde{\Gamma}}$, then Γ and $\tilde{\Gamma}$ are TPC-combinatorially related.*

From this, we may conclude that the diffeomorphism type of a divisor boundary determines the isomorphism type of its plumbing graph up to topological plumbing calculus. To prove this theorem, Neumann defines what it means for a plumbing graph to be in *normal form* and first shows that every plumbing graph may be put into normal form via topological plumbing calculus. Neumann then shows that the diffeomorphism type of plumbing boundaries whose graphs are in normal form do determine the isomorphism type of its associated graph. In order to write the definition of Neumann's normal form, it helps to discuss a special class of subgraphs relevant to the theorem: chains of spheres.

2.5 Chains of Spheres

A set of plumbing graphs important for our discussion are linear plumbings of spheres or "chains."

Definition 6. Let Γ be a plumbing graph. A *chain* is a subgraph $C \subset \Gamma$ of the form



with each vertex having degree at most 2 in Γ . The integers $-m_i$ are called *the components of the chain C* . A chain is called *maximal* if it is not contained in a strictly larger chain.

Remark 3. Chains are symmetric and so Y_C is diffeomorphic to $Y_{C'}$ where C' is the chain with components $(-m_{\ell}, \dots, -m_1)$.

The following lemma is a classical result in 3-manifold topology. There are many proofs of this result in the literature, see for example [Sym03].

Lemma 1. *Let $C \subset \Gamma$ be a chain with components $(-m_1, \dots, -m_{\ell})$ and $m_{\ell} \neq 0, 1$. Then Y_C is diffeomorphic to a Lens space $L(p, q)$ where*

$$-\frac{p}{q} = m_1 - \frac{1}{m_2 - \frac{1}{\ddots - \frac{1}{m_{\ell}}}}. \quad (2.3)$$

A chain is *normal* or *in normal form* if $m_i \geq 2$ for $i > 1$ and m_1 is either 0 or ≥ 2 . Chains that are in normal form define unique associated continued fractions and thus a unique associated lens space. This is the key property Neumann uses to prove Theorem 2.4.

From now on, we introduce the following notational conventions:

- We let $[m_1, \dots, m_\ell]$ denote the finite continued fraction with components $(m_1, \dots, m_\ell)$ as in the right-hand side of 2.3.
- We let $C(m_1, \dots, m_\ell)$ the chain with components $(-m_1, \dots, -m_n)$.
- We let $L(m_1, \dots, m_\ell)$ denote the lens space $L(p, q)$ with $-p/q = [m_1, \dots, m_\ell]$.

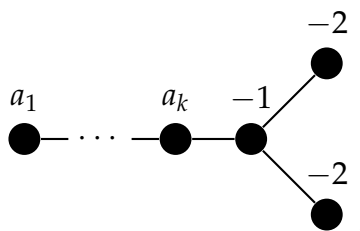
Each positive rational number r has a unique normal chain $C(m_1, \dots, m_\ell)$ associated to it where $r = [m_1, \dots, m_\ell]$. This follows directly from the fact that continued fraction expansions of rational numbers are unique if all the components m_i for $i > 1$ satisfy $m_i \geq 2$. Thus every normalized chain is homomorphic to a lens space $L(p, q)$ where p, q are relatively prime and $-p/q = r$.

2.6 TPC normal form

We are now able to define Neumann's normal form.

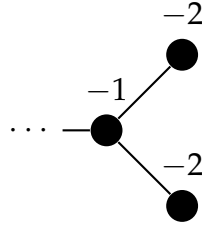
Definition 7 ([Neu81], Section 4). We say that a plumbing graph Γ is *in TPC normal form* if the following criteria are met:

1. No operation from topological plumbing calculus may be performed, except that Γ may have a component of the form:

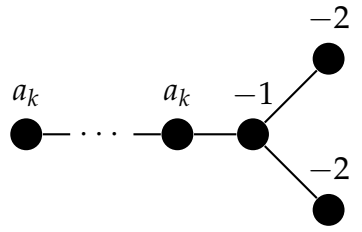


2. The self intersection numbers e_i associated to chains of Γ satisfy $e_i \leq -2$.

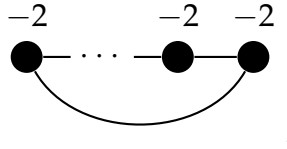
3. No portion of the graph has the form



unless it is in a component of Γ of the form



4. No component of Γ is isomorphic to:



In the context of divisor graphs, (4) is a special case of a *circular spherical divisor graph*. Such divisors and their graphs were classified in their entirety in [LM] and are not relevant to this paper.

Neumann's major results about graphs in normal form result in a proof of

Theorem 2.4 ([Neu81], **Theorem 3.1**). *If Y_D is diffeomorphic to $Y_{\tilde{D}}$, then Γ_D and $\Gamma_{\tilde{D}}$ are TPC-combinatorially related.*

It is first proven that: [[Neu81], **Theorem 4.1**] Any plumbing graph can be reduced to normal form using topological plumbing calculus. One then only needs to show:

Theorem 2.5 ([Neu81], **Theorem 4.2**). *Let $\Gamma, \tilde{\Gamma}$ be two plumbing graphs in topological normal form. Then Y_Γ is diffeomorphic to $Y_{\tilde{\Gamma}}$ if and only if Γ and $\tilde{\Gamma}$ are isomorphic.*

To prove Section 2.6, Neumann outlines a procedure for using the topological plumbing calculus to normalize a plumbing graph. Crucial to this procedure is an algorithm for normalizing chains. As we will see in the next section, this algorithm cannot be followed directly in the contact setting while preserving the contactomorphism type of the divisor boundary.

3 Contact Plumbing Calculus

We will see that not every move in the topological plumbing calculus respects the contact topology of the boundary. After determining which moves in the calculus are capable of being carried out in the contact setting, we will augment the calculus with new moves. These moves indeed are already possible within Neumann's calculus but nonetheless have their own unique usage which warrants their own names.

3.1 Contact-Sensitive TPC Moves

Theorem 2.4 tells us that if Y_D and $Y_{\tilde{D}}$ are diffeomorphic, then their divisor graphs are TPC-related. Since contactomorphism implies diffeomorphism, this result clearly also holds with respect to contactomorphism. The issue is that, while all associated topological constructions in Neumann's topological plumbing calculus preserve the diffeomorphism type of the divisor boundary, the moves may change the contactomorphism type of a divisor boundary. An example of a move that is not contact-preserving is the *interior/exterior +1 blow-up/blow-down* ([Neu81], Move 1) defined by

$$\dots \text{---} \overset{(g_1, k_1)}{\bullet} \text{---} \overset{(g_2, k_2)}{\bullet} \text{---} \dots \quad \longrightarrow \quad \dots \text{---} \overset{(g_1, k_1 + 1)}{\bullet} \text{---} \overset{1}{\bullet} \text{---} \overset{(g_2, k_2 + 1)}{\bullet} \text{---} \dots$$

or

$$\dots \text{---} \overset{(g, k)}{\bullet} \text{---} \longrightarrow \dots \text{---} \overset{(g, k + 1)}{\bullet} \text{---} \overset{1}{\bullet} \text{---} \dots$$

The following example demonstrating this was first given in [LM19] Example 2.21:

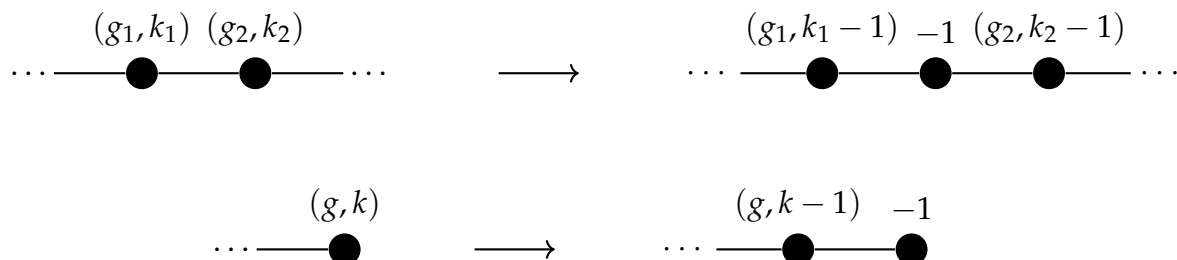
$$\overset{1}{\bullet} \text{---} \overset{2}{\bullet}$$

which is the $+1$ blow-up of



The contact 3-manifold associated to the second graph is (S^3, ξ_{std}) but the contact structure associated to the first is overtwisted.

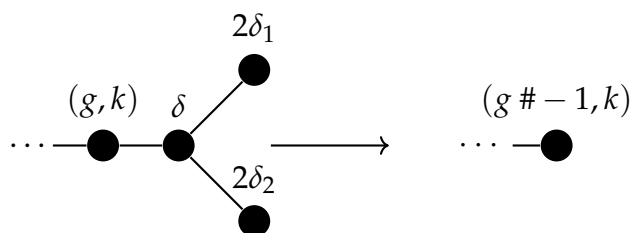
The *interior/exterior -1 blow-up/blow-down* defined by



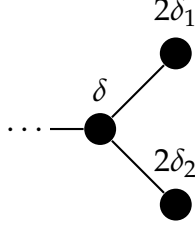
is contact-preserving. This was shown in [LM] using methods from toric geometry. Their construction exploits the toric structure of a neighborhood of the intersection of two smooth divisors. A toric diagram for a portion of a concave neighborhood can be obtained near the intersection points and the -1 blow-up can be performed in the toric region by modifying this diagram. One can then glue in the 4-manifold associated to the modified diagram by cutting out the toric region in the original neighborhood. This can be done while preserving concavity to yield a concave neighborhood of the symplectic divisor obtained after performing the blow-up.

Additionally, many of the moves outlined in topological plumbing calculus are not-applicable as they involve vertices with non-orientable bases which have no direct symplectic analog.

Every component of an SNC^+ divisor is orientable and so any of the moves involving non-orientable surfaces ([Neu81], Moves 2 and 4) can be completely removed from consideration with the exception $\mathbb{R}P^2$ -absorption:



Here $\delta_i = \pm 1$, $\delta = \frac{\delta_1 + \delta_2}{2}$, and the vertex with decoration $(g \# - 1, k)$ denotes the S^1 bundle over $\Sigma_g \# \mathbb{RP}^2$ with Euler number k . This move has no clear symplectic analog but subtlety appears in this story due to the simple fact that the manifold obtained from



by cutting at the plumbing torus corresponding to the leftmost edge is diffeomorphic to an orientable tubular neighborhood of a Klein bottle. This subtlety will be expanded upon in Section 3.5.

Edges from a vertex to itself never appear in SNC^+ plumbing graphs by the *simplicity* condition (i.e. all the components of D must be smooth manifolds). This allows us to discard any moves involving loops ([Neu81], Move 7). Since all intersections of SNC^+ divisors are positive, edge signs do not appear and so any moves involving edge signs can be discarded as well ([Neu81], Moves 3, 5, and 7). Finally, all of our contact 3-manifolds are without boundary so we may discard any moves dealing with graphs with non-empty boundary (Moves 6 and 8). Thus, the only move from Neumann's calculus that is applicable and contact-preserving is the -1 blow-up/blow-down.

3.2 Contact Plumbing Calculus

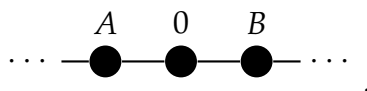
The discussion in the previous section tells us that we are disarmed considerably in the contact setting. This makes proving an analog of Section 2.6 more difficult since it is unclear whether one can reduce divisor graphs to their normal form with -1 blow-ups alone.

Fortunately, since many of Neumann's moves are not-applicable, we merely have to avoid using the $+1$ blow-up/blow-down or $\mathbb{RP}^2$ -absorptions to reduce to normal form. The proof of Section 2.6 ([Neu81], Theorem 4.1) crucially uses $+1$ blow-downs to normalize chains and so we must find another way to normalize chains in a way that preserves the contact structure of the divisor boundary. It turns out that -1 blow-ups are enough to get pretty far in the normalization process and draw some conclusions.

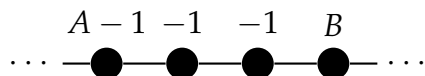
It is convenient to introduce a few additional contact-preserving moves that

we can work with. These moves follow directly from applications of -1 blow-ups and blow-downs but it is convenient to give them their own names:

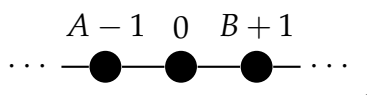
Construction 1 (0-curve transfer). Consider the following situation



We may blow up either edge connected to the 0-curve. For example, we may blow up the left edge



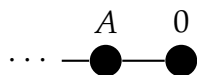
then blow down the (-1) -curve next to the B -curve to arrive at



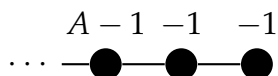
Effectively, we have taken from the A -curve and added to the B -curve. This process is called *0-curve transfer*. It's clear that a similar process may be used to transfer from the B -curve to the A -curve. This process can be iterated to increase the transfer between the chains. We will denote transfers with a directed edge



A special case is the following



in which we may blow up the internal edge



and blow down to

$$\dots \overset{A-1}{\bullet} \overset{0}{\bullet}$$

or we may blow up the exterior

$$\dots \overset{A}{\bullet} \overset{-1}{\bullet} \overset{-1}{\bullet}$$

and blow down to

$$\dots \overset{A+1}{\bullet} \overset{0}{\bullet}.$$

This is referred to as *transferring to/from nowhere* and will be denoted by

$$\begin{array}{ccc} \dots \overset{A}{\bullet} \overset{0}{\bullet} & \rightarrow & \dots \overset{A-k}{\bullet} \overset{0}{\bullet} \\ \text{curved arrow from } \overset{A}{\bullet} \text{ to } \overset{0}{\bullet} \text{ with box } +k & & \\ \dots \overset{A}{\bullet} \overset{0}{\bullet} & \rightarrow & \dots \overset{A+k}{\bullet} \overset{0}{\bullet} \\ \text{curved arrow from } \overset{0}{\bullet} \text{ to } \overset{A}{\bullet} \text{ with box } +k & & \end{array}.$$

Construction 2 (Sliding). Consider the following situation

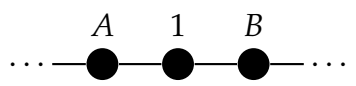
$$\dots \overset{A}{\bullet} \overset{0}{\bullet} \overset{0}{\bullet} \overset{B}{\bullet} \dots$$

We can use the left 0-curve in the 0-0 configuration to transfer all of the A -curve over. We arrive at

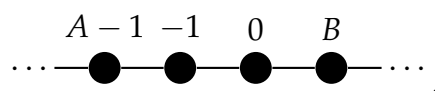
$$\dots \overset{0}{\bullet} \overset{0}{\bullet} \overset{A}{\bullet} \overset{B}{\bullet} \dots$$

Thus we have moved the 0-0 configuration to the left. This is called *sliding*. It's clear that we can slide in the other direction as well.

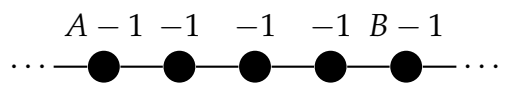
Construction 3 (Chain Replacement). We start with



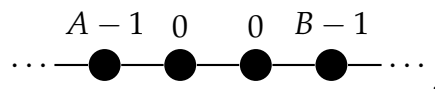
We blow up once on the left to get



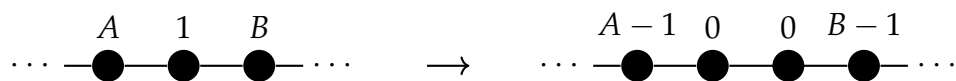
and once on the right to get



We then blow down the middle curve to arrive at



We refer to this process as *chain replacement* and we denote it by

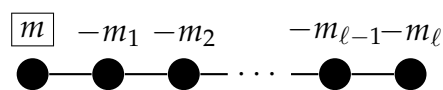


We will use these moves in order to perform a modified chain reduction procedure.

3.3 The Modified Chain Reduction Lemma

We may now prove the following key lemma:

Lemma 2. *Let Γ be a concave divisor graph. Then we may perform a finite sequence of combinatorial (-1) -blow-ups and (-1) -blow-downs on Γ to arrive at a graph $\Gamma^{\tilde{\zeta}}$ whose maximal chains are all of the form*



Where the m -vertex represents a linear chain of 0-curves of length $m \geq 0$ and $m_i \geq 2$. Such a chain is said to be in contact normal form. We call $\Gamma^{\tilde{\zeta}}$ the contact chain reduction of Γ .

Proof. Let $C \subset \Gamma$ be a maximal chain. Fix some linear ordering on the components of C so that $C = C(m'_1, \dots, m'_\ell)$. For exterior maximal chains, we always take m'_ℓ to be the decoration on the exterior-most curve. We progress according to the following procedure:

1. Blow down as many -1 -curves as possible from left to right (possibly in multiple stages). The result should be a maximal chain with no -1 -curves. If the resulting chain is normalized, proceed to the next chain.
2. For each configuration of the form

$$\dots \text{---} \overset{k_1}{\bullet} \text{---} \overset{0}{\bullet} \text{---} \overset{k_2}{\bullet} \text{---} \dots ,$$

transfer all of k_1 over the 0 -curve to the right

$$\begin{array}{ccc} \dots \text{---} \overset{k_1}{\bullet} \text{---} \overset{0}{\bullet} \text{---} \overset{k_2}{\bullet} \text{---} \dots & \longrightarrow & \dots \text{---} \overset{0}{\bullet} \text{---} \overset{0}{\bullet} \text{---} \overset{k_1+k_2}{\bullet} \text{---} \dots \\ & & \text{+}k_1 \end{array}$$

Now shift the 0 - 0 configuration all the way to the left.

3. Repeat steps (1) and (2), until the chain has no (-1) -curves and no 0 -curves except those on the left. Since both steps either preserve or reduce the number of vertices in the chain, this process will eventually halt.
4. The chain is now of the form:

$$\boxed{m} \text{---} \bullet \text{---} \mathcal{C}_+$$

where the $\boxed{m}$ -vertex is a linear chain 0 -curves of length m and $\mathcal{C}_+$ is a sub-chain without -1 -curves or 0 -curves. Starting from the left, identify the first k -curve in the chain with $k > 0$. The chain thus may be decomposed as

$$\boxed{m} \text{---} \bullet \text{---} \mathcal{C}_0 \text{---} \overset{k}{\bullet} \text{---} \mathcal{C}_+$$

where $\mathcal{C}_0$ is a chain with all decorations ≤ -2 and $\mathcal{C}_+$ is the rest of the chain. We could have $\mathcal{C}_0 = \emptyset$ and be in the situation:

$$\cdots \overset{0}{\bullet} \overset{k}{\bullet} \cdots$$

If the number of zeros on the left is odd, we simply use the chain of 0-curves to the left to transfer k into the rest of the graph. This increases the number of 0-curves on the left by one.

If the number of 0-curves on the left is even and non-zero, we may transfer $+2$ from the 0-curve left of the k -curve into the graph using the odd number of 0-curves to its left. This reduces the scenario to the case $\mathcal{C}_0 \neq \emptyset$ and decreases the number of 0-curves on the left by one.

In all other cases, we blow up the k -curve on the left $(k-1)$ times so that it becomes a zero curve, the chain is now of the form:

$$\boxed{m} \quad \bullet \text{---} \mathcal{C}_0 \text{---} \overset{-2}{\bullet} \overset{-2}{\bullet} \cdots \overset{-2}{\bullet} \overset{-1}{\bullet} \overset{1}{\bullet} \text{---} \mathcal{C}_+$$

and this process has added $(k-1)$ vertices. We then perform a chain replacement and replace the $+1$ -curve with a 0-0 configuration which adds an additional vertex. If $\mathcal{C}_+ \neq \emptyset$, then this leaves the chain in the form

$$\boxed{m} \quad \bullet \text{---} \mathcal{C}_0 \text{---} \overset{-2}{\bullet} \overset{-2}{\bullet} \cdots \overset{-2}{\bullet} \overset{-2}{\bullet} \overset{0}{\bullet} \overset{0}{\bullet} \overset{\tilde{k}}{\bullet} \text{---} \mathcal{C}_+$$

where we have

$$\tilde{k} = \begin{cases} \leq -2 \\ -1 \\ 0 \\ \geq 1 \end{cases}.$$

We may slide the 0-0 configuration so that it is of the form

$$\boxed{m+2} \quad \bullet \text{---} \mathcal{C}_0 \text{---} \overset{-2}{\bullet} \overset{-2}{\bullet} \cdots \overset{-2}{\bullet} \overset{-2}{\bullet} \overset{\tilde{k}}{\bullet} \text{---} \mathcal{C}_+$$

If $\mathcal{C}_+ = \emptyset$, then we proceed to the next chain after sliding. It will be important to notate the chain of -2 's for the next step. We will represent our chain by

$$\boxed{m+2} \quad \bullet - \mathcal{C}_0 - \boxed{(k-1)} - \overset{\sim k}{\bullet} - \mathcal{C}_+$$

By considering the -2 -curves at the tail end of $\mathcal{C}_0$, we may further write

$$\boxed{m+2} \quad \bullet - \mathcal{C}_- - \boxed{N} - \overset{\sim k}{\bullet} - \mathcal{C}_+$$

where N is a maximal subchain of -2 -curves and $\mathcal{C}_-$ is a subchain with only negative vertices and no -1 -curves.

5. If the algorithm did not terminate at the previous step, then we are in the following situation:

$$\boxed{M} \quad \bullet - \mathcal{C}_- - \boxed{N} - \overset{\sim k}{\bullet} - \mathcal{C}_+$$

We consider all cases for the value of $\tilde{k}$ separately:

- $\tilde{k} \leq -2$ or $\tilde{k} \geq 1$: For these, we continue to step through the chain or repeat the process outlined in step 4 if necessary.
- $\tilde{k} = 0$: We proceed as in step 2. At the end of this process, the chain is of the form:

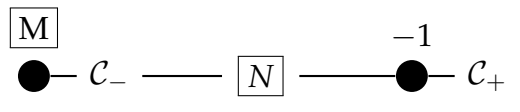
$$\boxed{M+2} \quad \bullet - \mathcal{C}_- - \boxed{N-1} - \mathcal{C}_+$$

where $\mathcal{C}_+$ is the remainder of the chain. If $\mathcal{C}_+$ is not empty, then the situation is like

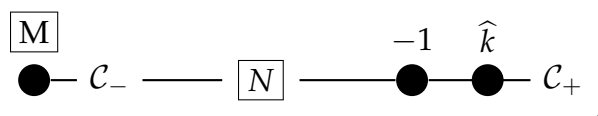
$$\boxed{M+2} \quad \bullet - \mathcal{C}_- - \boxed{N-1} - \overset{\hat{k}}{\bullet} - \mathcal{C}_+$$

so we may repeat this step for the new value $\widehat{k}$.

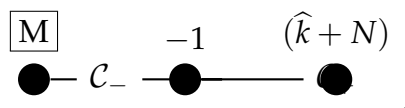
- $\widetilde{k} = -1$: We are in the situation



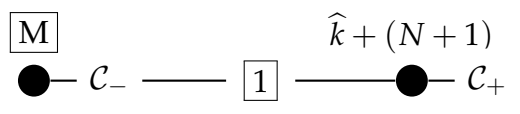
If $C_+ \neq \emptyset$, then we're actually in the situation



In this case (and even in the case when $C_+ = \emptyset$) we blow down the -1 -curve. Because of the chain of -2 's to the left of the -1 -curve, we may continue to blow down until the chain is of the form

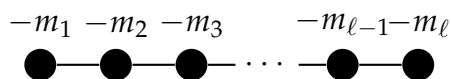


Blowing down the remaining -1 curve will affect the rightmost vertex in the subchain C_- . At worst, this vertex is a -3 -curve so that blowing down leave a trailing -2 -curve. So, at worst, the situation is

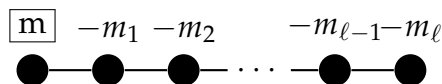


We continue to step through the chain, starting with the $\widehat{k} + (N + 1)$ curve.

We continue to repeat the present step until the maximal chain is in contact normal form.



or



where the $\boxed{m}$ -vertex is a linear chain of m 0-curves and $m_i \geq 2$.

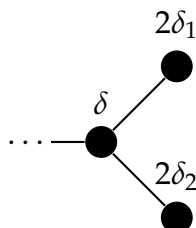
We then repeat the entire procedure for all other chains until all chains are in contact normal form. $\square$

3.4 Contact normal form plumbing graphs

The modified chain reduction lemma shows that we can almost fully normalize a chain in the contact setting using only blow-ups and blow-downs. The only difference between a fully normalized chain and a chain in contact normal form is the admissibility of any number of leading 0-curves. This difference, as well as the issues involving Klein bottle pieces discussed earlier, are the only obstructions to carrying out Neumann's reduction procedure in full. We make the following definition:

Definition 8. We say that a divisor graph Γ is in *contact normal form* if it is in TPC normal form except that:

- Γ may contain configurations of the form



where $\delta_i = \pm 1$, $\delta = \frac{\delta_1 + \delta_2}{2}$.

- A chain $C \subset \Gamma$ may contain any number of leading 0-curves.

With this definition and the discussion in the previous section, we have shown

Proposition 2. The contact chain reduction Γ^{ζ} of a plumbing graph Γ is in contact normal form.

We refer to Γ^{ζ} as the *contact reduction* of the graph Γ .

Corollary 1. *Any plumbing graph Γ can be reduced to a graph $\Gamma^{\tilde{c}}$ in contact normal form.*

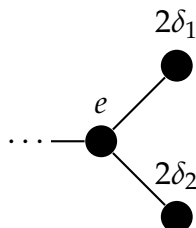
The two distinguishing features of a plumbing graph in contact normal form and a plumbing graph in topological normal form are:

- (i) The plumbing graph may have Klein bottle pieces.
- (ii) Any chains may contain any number of leading 0-curves.

The first of these conditions has topological implications and the second condition has symplectic and contact topological implications. In the next section, we will analyze the topological ramifications of having an embedded Klein bottle in a divisor boundary. The existence of leading zero chains is closely related to the concept of Giroux torsion from contact geometry. We hope to explicate the details of this in later work. For our applications, it suffices for us to understand the second condition in the context of our particular example (see Section 5). We start with the first condition.

3.5 Klein bottle pieces

We turn to the case of plumbing graphs which contain sub-graphs of the form



with $\delta_i = \pm 1$. If we cut Y_Γ along the plumbing torus corresponding to the leftmost edge of this sub-graph, we split Y_Γ into two pieces:

- A piece Y_Γ^* corresponding to the complement of the Klein bottle piece (i.e. the rest of the graph)
- A piece Y_K^* which is diffeomorphic to an orientable tubular neighborhood of a Klein bottle $K \subset Y_K^* \subset Y_\Gamma$.

To see that Y_K^* indeed has this topology, we observe from the corresponding portion of the plumbing graph that Y_K^* is a Seifert-fibered space over a disk with

two singular fibers with rational surgery coefficients $\pm\frac{1}{2}$. This gives a Seifert-fibered presentation of the twisted I -bundle over the Klein bottle (which itself is the twisted S^1 -bundle over S^1). (cf. [Hat], Chapter 2.1).

The boundary ∂Y_K^* is diffeomorphic to a torus. We say that K is *virtually compressible* if ∂Y_K^* is compressible, otherwise we say that K is *virtually incompressible* or *virtually essential*. For prime 3-manifolds, admitting an embedded virtually compressible Klein bottle puts severe restrictions on their topology.

Proposition 3. *Suppose that Y is a prime 3-manifold which contains an embedded Klein bottle. If this Klein bottle is virtually compressible, then Y may be given the structure of a Seifert-fibered space over $\mathbb{RP}^2$ with at most one singular fiber.*

Proof. As above, every Klein bottle piece yields an associated 3-manifold Y_K^* diffeomorphic to an orientable tubular neighborhood of an embedded Klein bottle with ∂Y_K^* diffeomorphic to a torus.

Let $\Delta \subset Y - Y_K^*$ be a compressing disk for ∂Y_K^* . The union $\partial Y_K^* \cup \Delta$ is a torus with a meridional disk attached. It follows that the boundary of a smoothing Y_K of the union of Y_K^* and a neighborhood of Δ is diffeomorphic to a sphere. This allows us to decompose Y_Γ as a connected sum with $Y_K \cup B^3$ appearing as a (non-trivial) summand. Since Y_Γ was assumed to be prime, we know that the complement $Y - Y_K$ is diffeomorphic to a 3-ball.

It follows that $Y - Y_K^*$ is diffeomorphic to a solid torus and so Y is diffeomorphic to a solid torus glued to an orientable tubular neighborhood of a Klein bottle along some diffeomorphism of their torus boundaries. Since Y_K^* has the structure of an S^1 -bundle over the Möbius strip, it follows that Y may be given the structure of a Seifert fibration over $\mathbb{RP}^2$ with at most one singular fiber (c.f. [Hat], Theorem 2.3(d)). $\square$

Corollary 2. *If Y_Γ is a prime plumbing boundary which is not Seifert-fibered, then every Klein bottle piece in Y_Γ contains a virtually essential Klein bottle.*

Thus if a plumbing boundary is not Seifert-fibered, it cannot contain a virtually compressible Klein bottle. While this does not fully deal with the issue of Klein bottle pieces, it does give rise to a simple test for eliminating the possibility of some embedded Klein bottles. To fully manage Klein bottle pieces in the contact setting, one would need to understand the contact-topological implications of the existence of virtually incompressible Klein bottles. We hope to pursue this in future work. For our present purposes, we will avoid analyzing this issue entirely by showing that the divisor boundary in our example cannot admit Klein bottles of any kind, virtually compressible or otherwise. This will be expanded upon in Section 4.

4 Main Results and Conclusions

In this section, we will collect a number of topological implications for divisor boundaries, compactifications, and neighborhoods that follow from the results in the previous sections. These results are collected with a goal of focusing on the nature of the structure of all divisor compactifications of a given Liouville 4-manifold X . We hope that these results will help contribute to the theory of *birational symplectic geometry* in dimension 4.

In the next section, as an application of these results, we will prove Theorem 1.1 which states that the subvariety $X_{KT} \subset M_{KT}$ does not admit a Stein completion to an affine variety. We will discuss how our arguments may be extended to produce other examples and conclude after briefly mentioning further directions for studying non-affine symplectic manifolds.

4.1 Structural theorems and topological implications

Let $(\bar{X}, d\lambda)$ be a Liouville domain with finite topology. We will consider the collection of concave SNC^+ divisor compactifications of $\bar{X}$. Recall that such a compactification is a pair (M, D) with (M, ω) a closed symplectic manifold and $D \subset M$ a concave SNC^+ divisor such that $M - D$ is deformation equivalent to the completion of $\bar{X}$. Each such compactification has an associated contact divisor boundary (Y_D, ξ_D) obtained as the contact-type boundary of a concave neighborhood (N_D, ω) of D . Recall that we say that a divisor D is obstructed if $\Gamma_D^\xi \not\approx \Gamma_D^{Top}$ and unobstructed otherwise. Our results allow us to characterize the compactifications of obstructed divisors whose associated divisor boundary Y_D is a prime 3-manifold. If this is the case, then we say that D is *prime*.

Theorem 4.1. *Let (M, D) be a concave SNC^+ divisor compactification of $\bar{X}$. Then if D is obstructed and prime, we have one or more of the following:*

- (i) *M is a blow-up of a rational or ruled symplectic manifold,*
- (ii) *the contact 3-manifold $(\partial\bar{X}, \xi)$ is Seifert-fibered over $\mathbb{RP}^2$ via a fibration with at most one singular fiber, or*
- (iii) *the contact 3-manifold $(\partial\bar{X}, \xi)$ contains an embedded incompressible Klein bottle.*

This characterization follows directly from a characterization of concave neighborhoods of obstructed prime divisors.

Theorem 1.3. *Let (N_D, ω) be a concave neighborhood of $D \subset M$. If the divisor D is obstructed and prime, then one or more of the following are true:*

- (i) *up to blow-ups (N_D, ω) is diffeomorphic to a concave divisor neighborhood $(N_{\tilde{D}}, \tilde{\omega})$ whose associated divisor contains a 0-curve S such that $S \cdot S = 0$,*
- (ii) *the contact boundary $(\partial N_D, \xi_D)$ is Seifert-fibered over the $\mathbb{RP}^2$ via a fibration with at most one singular fiber, or*
- (iii) *the contact boundary $(\partial N_D, \xi_D)$ contains an embedded incompressible Klein bottle.*

Proof. Since D is obstructed, $\Gamma_D^{\tilde{\xi}} \neq \Gamma_D^{\text{Top}}$. For this to occur, we must have one or both of the following:

- The contact reduction procedure for Γ_D produces chains with leading 0-curves,
- The contact reduction $\Gamma_D^{\tilde{\xi}}$ contains Klein bottle pieces.

By following the contact reduction procedure for Γ_D , we may perform a sequence of symplectic blow-ups on (N_D, ω) to obtain a concave divisor neighborhood $(N_{\tilde{D}}, \tilde{\omega})$ associated to a concave divisor $\tilde{D}$ whose associated graph $\Gamma_{\tilde{D}}$ is isomorphic to the contact reduction $\Gamma_D^{\tilde{\xi}}$.

If the first case occurs, then we may conclude that $N_{\tilde{D}}$ contains a symplectic sphere with self intersection number 0 and so we may conclude (i). If the second case occurs, since D is prime, conclusions (ii) and (iii) follow from Proposition 3 and Corollary 2. $\square$

Proof of 4.1. This result follows after applying a theorem of McDuff for ruled symplectic manifolds [McD90, Corollary 1.5(ii)]. In particular, Theorem 4.1(i) follows from the fact that a compact symplectic manifold with a symplectic sphere of self-intersection number 0 is the blow-up of a ruled surface (topologically an S^2 bundle over a closed surface). $\square$

Remark 4. If one removes the word “prime” from each of the above theorems, then conclusions (ii) and (iii) become less refined and we may only conclude that ∂N_D (or equivalently, the boundary $\partial \bar{X}$ of the Liouville domain) contains an embedded Klein bottle.

In the unobstructed case, the diffeomorphism type of the contact boundary is sufficient to characterize all concave divisor neighborhoods via direct application of [Neu81].

Theorem 1.5. Let N_D and $N_{\tilde{D}}$ denote concave divisor neighborhoods associated to a pair of unobstructed divisors D and $\tilde{D}$. Then if their divisor boundaries ∂N_D and $\partial N_{\tilde{D}}$ are diffeomorphic, the manifolds N_D and $N_{\tilde{D}}$ are diffeomorphic up to blow-ups. $\square$

The combined results about obstructed prime divisors and unobstructed divisors give a complete topological classification of neighborhoods of concave SNC^+ divisors.

With additional consideration, we may apply this to produce our most general result for divisor compactifications. As discussed, the divisor compactification (M, D) admits a decomposition of the form

$$M = \overline{X} \cup_{\Phi} N_D$$

where $(\overline{X}, d\lambda)$ is a Liouville domain, (N_D, ω) is a concave neighborhood of D and $\Phi: (\partial \overline{X}, \xi) \rightarrow (\partial N_D, \xi)$ is a contactomorphism defining the gluing. We may consider Φ to be an element of $\text{Cont}(\partial \overline{X}, \xi)$, the contactomorphism group of the contact boundary of $\overline{X}$. The isotopy class $[\Phi] \in \pi_0(\text{Diff}(\partial \overline{X}))$ determines the diffeomorphism type of M . This plus Theorem 1.5 is enough to conclude:

Theorem 1.6. Let D be a concave compactifying SNC^+ divisor for a 4-dimensional Liouville domain (X^4, ω) and let (M^4, ω) be a compactification of X by an SNC^+ divisor D with associated mapping class $[\Psi] \in \pi_0(\text{Cont}(\partial X))$ defining the capping. Then M either satisfies at least one of:

- (i) (M, ω) is a blow-up of a ruled symplectic manifold,
- (ii) ∂X is Seifert-fibered over the $\mathbb{R}P^2$ via a fibration with at most one singular fiber,
- (iii) ∂X contains a virtually essential Klein bottle

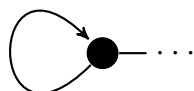
or any other SNC^+ divisor compactification $(\tilde{M}, \tilde{\omega})$ with the same mapping class $[\Psi]$ can be obtained from (M, ω) via blow-ups and blow-downs.

This result could be strengthened if one could understand to what extent the contact mapping class $[\Psi]$ may vary. For example, if every diffeomorphism $\Psi: \partial N_D \rightarrow \partial N_D$ extends to a diffeomorphism $\tilde{\Psi}: N_D \rightarrow N_D$, then the diffeomorphism type

of any divisor compactification of X is completely determined by the diffeomorphism type of X and the topology of D , independent of the choice of contactomorphism defining the gluing. We will use this to finish a proof of Theorem 1.1 in the next section.

4.2 0-spheres and Giroux torsion

We will only consider connected plumbing graphs Γ for the remainder of the discussion. For now, we also assume that Γ is without loops i.e. without configurations of the form



Let Y_Γ be the 3-manifold plumbing obtained from Γ . When Γ is the graph associated to a symplectic normal crossing divisor $D \subset M$, the positive (resp. negative) GS-criterion says that D has a concave (convex) regular neighborhood if and only if the linear equation

$$Q_D \underline{b} = \underline{a}$$

has a solution $\underline{b} \in \mathbb{R}^N$ with all positive (negative) entries. In both these cases, the concave/convex neighborhood furnishes the boundary plumbing with a natural contact structure which is co-oriented relative to the boundary orientation.

The anatomy of this contact structure is outlined in but briefly, the contact boundaries are built from pieces of two types:

- Type I Pieces: which are diffeomorphic to S^1 -bundles over surfaces with boundary.
- Type II Pieces: neighborhoods of the plumbing tori which are diffeomorphic to $T^2 \times I$ equipped with a minimally twisting contact structure.

The induced contact structures Type I pieces are regular in the sense of Boothby and Wang and so their Reeb foliations provide them with a natural S^1 -fibration structure. Type II pieces are simple interpolations between the characteristic slopes of the torus boundaries of two neighboring Type I pieces, connected according to

Γ_D . Without much additional work, one can construct similar contact structures on arbitrary plumbings.

Theorem 4.2. *Let Γ be a plumbing graph. Then there exists a contact structure ξ_Γ on the 3-manifold plumbing Y_Γ which is transverse to the S^1 -fibration away from the plumbing tori and minimally twisting near the plumbing tori. Any other contact structure on Y_Γ satisfying these conditions is contactomorphic to ξ_Γ . In the case when Γ satisfies the positive/negative GS-criterion, ξ_Γ is co-orientable and is contactomorphic to the natural contact structure induced by the concave/convex neighborhood.*

Proof. We proceed to build our contact manifolds out of two types of pieces as before:

- Type I pieces are diffeomorphic to S^1 -bundles over surfaces with boundary with horizontal contact structures.
- Type II Pieces are diffeomorphic to $T^2 \times I$ with minimally twisting contact structures.

We will have one Type I piece Y_v for each vertex v in Γ . The contact manifold Y_v will be obtained by removing solid tori from a closed contact manifold. That closed contact manifold will depend on the decorations (g, k) of the vertex v . We will take different approaches depending on whether $k = 0$ or $k \neq 0$. We will handle the case $k \neq 0$ first.

When D is a smooth divisor, the only entry in the intersection form Q_D is $k := D \cdot D$ and so the GS-criteria become equivalent to the requirement that the self-intersection number take a definite sign. In the case when $k > 0$, D admits a concave neighborhood. In the case $k < 0$, it admits a convex one.

In both cases, the induced contact manifold $(\partial N_D, \xi_D)$ admits the structure of a Boothby-Wang bundle over the divisor, i.e. a bundle map $\pi: (\partial N_D, \xi_D) \rightarrow (D, \omega)$ which satisfies $\pi^* d\alpha = \omega$. In this setting, the Reeb vector field is everywhere tangent to the S^1 -fibers and the bundle map π is obtained as a quotient of the periodic flow of the Reeb vector field. Because of the co-orientation of the contact structure, in both the $k > 0$ and $k < 0$ cases, the bundle is $\pi: (\partial N_D, \xi_D) \rightarrow (D, \omega)$ is the principal S^1 -bundle with Euler number $e(\partial N_D, \xi_D) = |k|$.

We start with the bundle $\pi: (\partial N_D, \xi_D) \rightarrow (D, \omega)$ and remove small solid tori neighborhoods of a collection of d_v disjoint fibers where d_v is the degree of the vertex. This leaves us with a contact S^1 -bundle over a surface with boundary with Reeb fibers. This piece has d_v boundary components, one for each solid torus that was removed.

Finally, there is the case when $k = 0$. If Σ is a closed surface, by [ET] and [Hon00] there is a unique contact structure on $\Sigma \times S^1$ which is C^0 -close to the foliation $\Sigma \times \{pt\}$. In the case when $g(\Sigma) > 0$, this contact structure is horizontal with respect to the natural S^1 -fibration structure. If $g(\Sigma) = 0$, this is the unique almost-horizontal contact structure on $S^2 \times S^1$. In either case, we start with $\Sigma \times S^1$ and remove small solid torus neighborhoods of a collection of d_v disjoint fibers for which the contact structure is horizontal nearby.

We construct pieces of Type II by interpolating between the characteristic slopes of the boundary tori of two neighborhood Type I pieces, connected according to Γ .

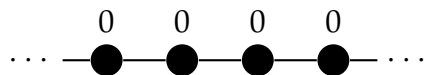
It is clear from construction that our contact structures agree with those induced by concave/convex neighborhoods of divisors. The fact that this contact structure is globally co-orientable follows from the co-orientability of the induced structures. □

We call any realization (Y_Γ, ξ_Γ) of this unique contactomorphism type a *contact plumbing*. Since the natural contact structures on concave divisors satisfy these hypotheses, we have:

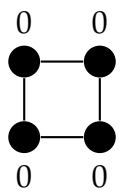
Corollary 3. *Any convex/concave plumbing boundary $(\partial N_D, \xi)$ is contactomorphic to the contact plumbing (Y_D, ξ_D) determined by the divisor graph Γ_D .*

We conclude that the contact geometry of concave divisor boundaries is topologically determined by the divisor graph Γ_D . We are thus free to draw conclusions about the contactomorphism type of a concave divisor boundary from topological information about the graph Γ_D . One such conclusion which one can reach is:

Theorem 4.3. *If (Y_Γ, ξ_Γ) is a contact plumbing and Γ contains a configuration of the form*

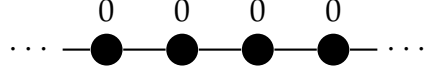


which is not part of a component of Γ of the form

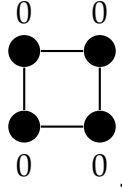


then (Y_Γ, ξ_Γ) has nontrivial Giroux torsion.

Proof. Any portion of a 3-manifold boundary corresponding to the configuration



admits a local toric contact structure. Each 0-framed sphere contributes $> \frac{\pi}{2}$ -torsion to the contact structure and so we will have $> 2\pi$ torsion except in the case when Γ_D is of the form



for which we have $Y_D \approx T^3$ and ξ_D contactomorphic to the universally tight contact structure on T^3 . $\square$

If a plumbing graph Γ admits such a configuration, we say that it has *positive torsion*. Any divisor D whose divisor graph Γ_D has positive torsion is also said to have positive torsion. This result has symplectic-topological implications. If (M, ω) is a closed symplectic 4-manifold and $D \subset M$ is a concave divisor, then any concave divisor neighborhood $N_D \subset M$ determines a weak filling $X = M - N_D$ of $(\partial N_D, \xi_D)$. Since Giroux torsion is an obstruction to weak fillability [Gay06], we immediately conclude:

Corollary 4. *If Γ is a plumbing graph with positive torsion, then Γ cannot appear as the divisor graph of an embedded concave divisor $D \subset M$. In other words, concave divisors with positive torsion cannot be embedded into closed symplectic 4-manifolds.*

By contrast, every convex symplectic divisor embeds into a closed symplectic 4-manifold since we may always cap off a convex divisor neighborhood.

5 Application: A non-algebraic Stein manifold

Let M_{KT} denote the Kodaira-Thurston manifold (see [Thu76]) defined as $M_{KT} := \mathcal{M}_\phi \times S^1$ where $\mathcal{M}_\phi$ is the mapping torus of a right Dehn twist $\phi: T^2 \rightarrow T^2$. We

give $\mathcal{M}_\phi$ local coordinates $(\theta_1, \theta_2, \theta_3)$ and so M_{KT} may be given local coordinates $(\theta_1, \theta_2, \theta_3, \theta_4)$ and we define a symplectic form

$$\omega := d\theta_1 \wedge d\theta_2 + d\theta_3 \wedge d\theta_4.$$

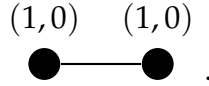
We note that (M_{KT}, ω) contains a natural SNC^+ divisor. Let $\sigma: S^1 \rightarrow \mathcal{M}_\phi$ be a section of the mapping torus and let $\pi: \mathcal{M}_\phi \rightarrow S^1$ be the natural projection map. We define two tori

$$D_L := \pi^{-1}(pt) \times \{pt\}$$

$$D_R := \sigma(S^1) \times S^1$$

and let $D := D_L \cup D_R$. Evidently, D_L and D_R are ω -orthogonal symplectic divisors and so D is an SNC^+ divisor.

The plumbing graph associated to the divisor D is



and so the intersection form is given by

$$Q_D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which admits many GS-positive solutions (regardless of the symplectic area chosen for D_L and D_R). It follows that N_D is concave and so it has contact-type boundary.

We let $X := M_{KT} - D$. Since X is a symplectic punctured torus bundle over a symplectic punctured torus, X is a Liouville manifold (in fact, a Stein domain) [BHM15, Theorem 1.2]. We claim

Theorem 1.1. *The Liouville manifold X is not biholomorphic to any affine variety.*

whose proof will rely on the main application of our result. We need several results in order to establish this theorem.

Proposition 4. *Any concave SNC^+ compactifying divisor $\tilde{D}$ for X_{KT} is unobstructed and prime.*

Recall that a divisor $\tilde{D}$ is unobstructed if $\Gamma_{\tilde{D}}^{\xi} \approx \Gamma_{\tilde{D}}^{Top}$ and “prime” simply means that each divisor boundary is a prime 3-manifold. The fact that all compactifying divisors for X are unobstructed allows us to apply Theorem 1.5. From there, we verify:

Proposition 5. *If $\tilde{D}$ is a compactifying divisor for X_{KT} , then Every diffeomorphism $\Psi: \partial N_{\tilde{D}} \rightarrow \partial N_{\tilde{D}}$ extends to a diffeomorphism $\tilde{\Psi}: N_{\tilde{D}} \rightarrow N_{\tilde{D}}$.*

By our previous discussion, may conclude:

Theorem 5.1. *Let $(\tilde{M}, \tilde{\omega})$ be a concave divisor compactification of X_{KT} . Then, up to blow-ups, M is diffeomorphic to M_{KT} .*

From these results, we may prove our main theorem:

Proof of Theorem 1.1. Suppose, on the contrary, that X is affine. Then it follows that it admits an algebraic SNC^+ compactification $(\tilde{M}, \tilde{D})$. By Proposition 4 we see that the divisors D and $\tilde{D}$ are both unobstructed. We may thus apply Theorem 1.5 and see that we may perform a sequence of blow-ups on $\tilde{M}$ until our divisor $\tilde{D}$ becomes diffeomorphic to D . This process results in a 4-manifold $\hat{M}$. It then follows from Proposition 5 that $\hat{M}$ is diffeomorphic to M_{KT} . This cannot possibly occur since $b_1(M_{KT}) = 3$ and blowing up does not alter the first Betti number. $\square$

We will need a few different sublemmas:

- (i) The divisor boundary Y_D is a prime 3-manifold
- (ii) Y_D does not contain any embedded Klein bottles
- (iii) X does not admit a compactification (M, D) such that D contains a sphere with self intersection number 0.

Together, items (i) and (ii) tell us that no divisor with divisor boundary diffeomorphic to Y_D can contain Klein bottle pieces. Item (iii) allow us to avoid issues involving 0-curves as they cannot be a component in any divisor compactification of X . In particular, this implies that no 0-curves appear in the contact chain reduction of any concave divisor graph with divisor boundary diffeomorphic to Y_D . From all of this, it will follow that the contact chain reduction of any such divisor graph must be isomorphic to its topological reduction which is the key to concluding our result. We will prove the lemmas listed above in sequence.

5.1 Prime divisor boundary

Lemma 3. *The manifold Y_D is prime.*

Proof. Suppose not and suppose we may compose Y_D as a non-trivial connected sum. Let $S \subset Y_D$ denote the embedded S^2 defining the connected sum decomposition. Recall that Y_D is the result of a gluing of two S^1 -bundles over punctured tori. We will let Y_L and Y_R denote these bundles. Let $T \subset Y_D$ be the plumbing torus, i.e. the image of the boundary tori of Y_L and Y_R under this gluing.

We wish to understand how S intersects T . After perturbing via an isotopy, we may assume that S and T intersect transversely. By compactness, we know that S and T intersect along a collection of closed curves. Removing T splits S into a collection of surfaces each having some number of boundary components, each of which is a curve in the collection. Since S is a sphere, one of these pieces must be a disk.

The plumbing torus T is an incompressible surface and so the boundary of this disk must also bound an embedded disk in T . This implies that each of the curves in our collection are contractible in T . Since Y_D is prime, these two disks bound a 3-ball and so we may use this 3-ball to push the disk across T and isotope it until it lies on only one side of the decomposition. We may repeat this argument for the remaining collection and eliminate every intersection, pushing everything to the same side of T . It follows that we may assume that S lies on one side of the plumbing torus or the other. This would allow us to decompose one of the sides as a non-trivial connected sum. Since both $\pi_2(Y_L) \approx \pi_2(Y_R) \approx 0$, we know this is impossible. $\square$

5.2 No Klein bottles

We will now prove that Y_D is free of Klein bottles.

Lemma 4. *The manifold Y_D contains no embedded Klein bottles.*

Proof. We start by showing that Y_D cannot contain compressible Klein bottles. By Lemma 3 and Proposition 3, if Y_D did contain a virtually compressible Klein bottle, then it may be given the structure of a Seifert fibration with at most one singular fiber. If this were the case, its fundamental group would admit the following presentation:

$$\pi_1(Y_D) = \langle \mu, \lambda : \mu\lambda\mu^{-1} = \lambda^{-1}, \mu^{2a} = \lambda^{-b} \rangle$$

for some $a, b \in \mathbb{Z}$.

We understand the topology of Y_D as the result of plumbing surgery on a pair of trivial S^1 -bundles over T^2 . From this description, we may give a presentation of the fundamental group (c.f. [Orl72, Chapter 5.3]):

$$\pi_1(Y_D) = \langle u, v, x, y, F_L, F_R : F_L = [x, y], F_R = [u, v] \rangle.$$

By abelianizing, we see that

$$H_1(Y_D) = \pi_1(Y_D)_{Ab} = \mathbb{Z}^4$$

so the abelianization of the second presentation is obviously not equal to the abelianization of the first. Thus we arrive at a contradiction.

We then turn to the case of embedded incompressible Klein bottles. Topologically, Y_D is the boundary of a plumbing of two trivial $\mathbb{D}^2$ -bundles over T^2 . We let $T \subset Y_D$ be a fixed representative of the sole plumbing torus. Suppose that $K \subset Y_D$ is an embedded virtually incompressible Klein bottle.

Cutting Y_D along T splits the manifold into two pieces Y_L and Y_R . Both of these manifolds are diffeomorphic to S^1 -bundles over a torus minus an open disk. This cut splits K into a collection $\{\Sigma_i\}_{i \in \mathcal{I}}$ of surfaces with boundary. A fixed boundary component of any one of these surfaces is some closed curve in T . Since K is an essential surface, each Σ_i is an incompressible and boundary incompressible surface in either Y_L or Y_R . These surfaces must be either horizontal or vertical with respect to the Seifert fibrations (c.f. [Hat], Proposition 1.12). Recall a surface Σ in a Seifert-fibered space Y with Seifert-fibration $\pi : Y \rightarrow S$ is *vertical* if Σ is a union of fibers of π and *horizontal* if it is transverse to the fibers of π . If Σ is horizontal, the restriction of π gives Σ the structure of a branched cover over S .

Since the plumbing torus T is incompressible, we can assume that none of the Σ_i are disks. Thus since K is broken into the pieces Σ_i by T , we know that each Σ_i must be either:

- a mobius band,
- a punctured Klein bottle,
- an annulus, or
- a Klein bottle.

If Σ_i is a horizontal surface, then it must be a branched cover over a punctured T^2 and hence it must be oriented and have negative Euler characteristic (by the

Riemann-Hurwitz formula, [Jos97, Theorem 2.5.2]). Thus none of the Σ_i are horizontal.

A vertical surface must be either an annulus or a torus so K is a union of vertical annuli in Y_L and Y_R . In order for these pieces to glue to form a Klein bottle, we must always have a pair of pieces $\Sigma_L \subset Y_L$ and $\Sigma_R \subset Y_R$ which are glued together in the union. Since the gluing between Y_L and Y_R sends the fiber of one side to a section on the other side, we cannot have that both of these surfaces are vertical, a contradiction. $\square$

5.3 No embeddings into ruled surfaces

In order to establish (iii) we need the following.

Lemma 5. *The cohomology group $H_2(X)$ contains a class with positive self intersection.*

Proof. We consider our chosen section $\sigma: S^1 \rightarrow \mathcal{M}_\phi$ of the mapping torus and we let $\pi: \mathcal{M}_\phi \rightarrow S^1$ be the natural projection. As above, we have

$$D_L := \pi^{-1}(pt) \times \{pt\}$$

and

$$D_R := \sigma(S^1) \times S^1.$$

We may trivialize the T^2 -bundle $\pi: M_{KT} \rightarrow T^2$ near the fiber D_L in order to obtain generators for nearby T^2 -fibers. Given $q \in S^1$ near our chosen point $p := \pi(D_L)$, we let D_q denote the fiber lying over it. We let $\mu_q, \lambda_q \in H_1(D_q)$ denote the generators obtained from the trivialization. We choose our trivialization such that μ_q is parallel to the curve along which we perform a Dehn twist in each mapping torus $\mathcal{M}_\phi \times \{pt\} \subset M_{KT}$.

We let $A_L \in H_2(M_{KT})$ denote the homology class represented by a product of λ_q with the circle $\{pt\} \times S^1 \subset M_{KT}$. We let $A_R \in H_2(M_{KT})$ denote the homology class represented by a product of a parallel copy of $\sigma(S^1) \times \{pt\}$ (i.e. one disjoint from D_R) with μ , the image of $\mu_q \times I$ in $M_\phi \subset M_{KT}$. We let $A = A_L + A_R$. Evidently, we have $A \cdot A > 0$ which completes the proof. $\square$

With this, we may prove:

Lemma 6. *X cannot admit a divisor compactification (M, D) such that D contains a sphere of self intersection number zero.*

Proof. Assume by way of contradiction that X admits such a compactification (M, D) . Let $S \subset D$ denote a sphere with $S \cdot S = 0$. By [McD90, Corollary 1.3(ii)], the manifold M is diffeomorphic to an S^2 -bundle over a closed surface Σ blown up some number of times and S is homologous to a fiber of this bundle. It is straightforward to compute the intersection form of this manifold:

Let $\tilde{M}$ be the minimal symplectic manifold obtained by blowing down all of the exceptional curves of M (so that $\tilde{M}$ is a genuine S^2 -bundle over a surface). The second homology group $H_2(\tilde{M}; \mathbb{Z})$ is generated by a generic fiber F and a generic section $\tilde{\Sigma} \subset \tilde{M}$. If $\tilde{M}$ is trivial, then the intersection form is given in the $\{[F], [\tilde{\Sigma}]\}$ -basis by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and if $\tilde{M}$ is non-trivial, the intersection form is given by

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

In order to obtain M from $\tilde{M}$, we must perform blow-ups at a collection of disjoint points $\{p_1, \dots, p_m\} \subset \tilde{M}$. We may assume these points are disjoint from F and $\tilde{\Sigma}$. After performing these blow-ups, the intersection form of M is given by either

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus -\mathbb{1}[m]$$

or

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \oplus -\mathbb{1}[m]$$

where $-\mathbb{1}[m]$ denotes an $m \times m$ matrix with -1 's along the diagonal and zeros off the diagonal.

We consider $Z = \tilde{M} - F$. Since F is parallel to the 0-framed sphere in D , we must have $X \subset Z$. We note that the intersection product

$$- \cdot - : H_2(Z) \otimes H_2(Z) \rightarrow \mathbb{Z}$$

is negative semi-definite. To see this, we simply observe that Z is diffeomorphic (via our generic section $\tilde{\Sigma}$) to a blow-up of $\tilde{\Sigma}^* \times S^2$ where $\tilde{\Sigma}^* = \tilde{\Sigma} - \{pt\}$. Every class $\beta \in H_2(Z)$ has $\beta \cdot \beta \leq 0$ so the induced intersection product $H_2(Z) \otimes H_2(Z) \rightarrow \mathbb{Z}$ is negative semi-definite. The pushforward of the class from Lemma 5 along the inclusion is a class with positive self intersection and so we have a contradiction. □

5.4 Mapping class considerations and conclusion

In order to apply the full strength of our main theorem as in the conclusion of the previous section, we prove:

Proposition 5. *Every diffeomorphism of Y_D extends to the neighborhood N_D i.e. for any diffeomorphism $\psi: Y_D \rightarrow Y_D$, there exists a diffeomorphism of $\Psi: N_D \rightarrow N_D$ such that $\Psi|_{Y_D} = \psi$.*

Proof. We note that Y_D is not a Seifert fibered space. This follows, for example, from a corollary of Neumann's theorem ([Neu81], Corollary 5.7) which states that the normal form of a plumbing graph associated to a Seifert fibered space must be star shaped (i.e. a single genus g component connected to a number of chains of spheres). Γ_D is in normal form but it is certainly not star shaped.

If $T \subset Y_D$ is a plumbing torus representative, then $Y_D - T$ is a pair of S^1 -bundles over surfaces with boundary as above. It follows that each piece in this decomposition is Seifert-fibered.

It follows that the singleton set $\{T\}$ yields a JSJ-decomposition of Y_D (c.f. [Hat], Chapter 2). By the uniqueness theorem for JSJ-decompositions, we may assume that ψ is isotopic to a diffeomorphism which fixes a regular neighborhood of T . Thus ψ is isotopic to a diffeomorphism of the two pieces Y_L and Y_R which is the identity near the boundary. This diffeomorphism respects the Seifert fibrations of the two pieces and so is induced by a pair of diffeomorphisms $f_L, f_R: T_*^2 \rightarrow T_*^2$ of the T_*^2 , the torus minus a disk. We may use these diffeomorphisms to define our diffeomorphism $\Psi: N_D \rightarrow N_D$ which completes the proof. $\square$

Once we know that every diffeomorphism extends, we may reach our strongest possible conclusion about the collection of all compactifications.

Theorem 5.2. *Let $(M, \tilde{\omega})$ be a concave divisor compactification of X_{KT} . Then, up to blow-ups, M is diffeomorphic to M_{KT} .*

Proof. Given such a compactification $(M, \tilde{\omega})$ by a divisor $\tilde{D}$, we let $\Gamma_{\tilde{D}}$ denote its associated divisor graph. Since M_{KT} and M are both divisor compactifications of X_{KT} , it follows that Y_D and $Y_{\tilde{D}}$ are both contactomorphic to the boundary of the Liouville subdomain $\bar{X} \subset X_{KT}$.

Since Y_D is unobstructed, the contact reductions Γ_D^{ξ} and $\Gamma_{\tilde{D}}^{\xi}$ are both isomorphic to their respective topological reductions. By Neumann's theorem (and the modified chain reduction lemma), it follows that we may relate the divisor graph

of Γ_D to $\Gamma_{\tilde{D}}$ via a sequence of blow-ups and blow-downs. From this, we may perform corresponding blow-ups on $(M, \tilde{\omega})$ in order to transform the divisor $\tilde{D}$ into D (this follows because Γ_D is already in topological normal form). This presents a blow-up $B\ell(M)$ as a compactification of X_{KT} by the same divisor D and so M and $B\ell(M)$ differ only by the isotopy class of diffeomorphism of Y_D defining these compactifications. By Proposition 5, there is only one such isotopy class. It follows that $B\ell(M)$ and M_{KT} are diffeomorphic $\square$

Thus we may conclude:

Theorem 1.1. *The manifold $X = M_{KT} - D$ is not biholomorphic to any affine variety.*

A similar argument should work for any Liouville manifold with the following assumptions:

1. X is presumed to be affine or quasi-projective.
2. The boundary $\partial\bar{X}$ of the Liouville subdomain $\bar{X} \subset X$, unique up to Liouville isomorphism, is prime.
3. All compactifying SNC^+ divisors of X are all unobstructed.
4. X admits a compactification M with $b_1(M)$ odd.

In light of [McL18] and [Sei08], this is the first example of a non-affine symplectic manifold whose obstruction from being affine cannot be detected by the growth rate of symplectic homology. The existence of this example tells us that there should be a deeper obstruction to being affine that assumes the same basic topological setup as above and is not captured via growth rate techniques. The existence and nature of such an obstruction is still currently unknown.

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