KÄHLER-RICCI FLOW IS SYMPLECTIC DEFLATION PLUS AN EXACT SYMPLECTIC DEFORMATION

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ABSTRACT. We demonstrate that Kähler-Ricci flow on a Fano surface decomposes as a symplectic deflation along a plurianticanonical divisor plus an exact symplectic deformation in the divisor complement. We apply this decomposition to the study of certain projective bundles. We establish the optimal decay rate in the setting of Song, Székelyhidi, and Weinkove for a family of toric metrics which match the cohomological profile of Kähler-Ricci flow. In doing so, we identify a C^2-obstruction to proving the optimal decay rate in the setting of ruled toric surfaces. We present these ideas as a new symplectic-topological approach to the study of Kähler-Ricci flow.

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1. Introduction

Suppose that (X, g, ω) is a Kähler manifold. The Kähler-Ricci flow equation is given by

$$\begin{cases} \partial_t \omega_t = -Ric_{\omega_t} \\ \omega_0 = \omega \end{cases} \tag{1}$$

Any 1-parameter family (X, g_t, ω_t) of Kähler manifolds satisfying this equation is called a *solution* to the Kähler-Ricci flow equation. Here Ric_{ω_t} is the Ricci form of the Kahler metric g_t defined by

$$Ric_{\omega_t} = -i\partial\overline{\partial}\log\left(\det g_{j\bar{k}}(t)\right)$$

Cohomologically

$$[Ric_{\omega_t}] = c_1(X)$$

and so we have an induced flow on cohomology

$$\begin{cases} \partial_t[\omega_t] = -c_1(X) \\ [\omega_0] = [\omega] \end{cases} \tag{2}$$

which has a unique solution $[\omega_t] = [\omega_0] - tc_1(X)$.

This observation leads one to conclude that for *Fano* manifolds, manifolds with $c_1(X) > 0$, one should expect a finite extinction time for X under the Kähler-Ricci flow. In 2003 through public communication during a visit to MIT (c.f. [ST08]), Perelman demonstrated that for a Fano manifold X, under the assumption that $[\omega_0] = Tc_1(X)$, one must have

$$diam_{\omega_t}(X) \leq C\sqrt{T-t} \tag{3}$$

where the left hand side is the diameter of X with respect to the Kähler metric ω_t. Sesum and Tian in [ST08] published a proof of Perelman's estimate. It was then conjectured by Tian [Tia08] and later proven by Song [Son14] that the condition $[\omega_0] = Tc_1(X)$ holds if and only if X is birationally Fano, so one should always expect such a decay rate in that setting. This fact and other related results [FIK03][TZ06][Tia08] have greatly contributed to the present day analytic minimal model program which attempts to center Kähler-Ricci flow as a process that tends to converge toward minimal birational models through singularities such as analytic blowdowns [SW13, SW14].

Perelman's diameter decay rate estimate has been shown to have a major impact on the strucuture of other complex manifolds under Kähler-Ricci flow. In [SW11], Song and Weinkove make advances toward systemetizing these Fano collapse results to Fano fibrations through their study of Hirzebruch surfaces. Subsequent work by [Fon14] on projective bundles continued this general direction. A major breakthrough came with [SSW13] in which they proved that a projective bundle $\pi\colon (X, \omega_0) \to (B, \omega_B)$ under the assumption that

$$[\omega_0] - Tc_1(X) = [\pi^*\omega_B]$$

admits the following estimate for the diameter of the fiber F

$$diam_{\omega_t}(F) \leq C(T-t)^{1/3}.$$

In their work, it is noted that this falls short of the expected optimum $\sqrt{T-t}$ bound. Several improvements were made in various settings and this thinking was applied to other

Fano fibrations as well [FZ17][She21] [CT15][TZ18]. In 2015, Fong achieved the optimal upper bound for the diameter of the fiber under the assumption of a Ricci curvature upper bound [Fon15]. For projective bundles, research developments culminated in Zhang-Zhang's 2023 improvement of the exponent to $1/2 - \varepsilon$ for every $\varepsilon > 0$, just shy of the optimum $1/2$ [ZZ23]. The author also notes very recent results due to Bednarek [Bed25], Jian-Song [JS26a][JS26b], and Xu-Zheng [XZ26] which achieve the optimum decay rate in various settings.

In this article, we use symplectic geometry to improve our understanding of the optimal collapse rate in the settings of toric ruled surfaces. We use our findings in order to show the following theorem in that context (c.f. [SSW13], Section 1).

Theorem 1.1. *Let (X, g, ω) be a toric ruled surface with compatible ruling. Let (g_t, ω_t) be the Kähler-Ricci flow on X starting at (g, ω) such that the volume of the fiber F collapses at time $T > 0$. Then there exists a 1-parameter family of toric Kähler metrics $\widetilde{\omega}_t$ such that*

$$\omega_t = \widetilde{\omega}_t + i\partial\overline{\partial}\psi_t$$

and we have the following estimate

$$\mathit{diam}_{\widetilde{\omega}_t}(F) \leq C(T-t)^{1/2}.$$

This establishes the sharp upper bound in the setting of Song-Székelyhidi-Weinkove in [SSW13] *for the family $\widetilde{\omega}_t$.*

Our proof of this result exploits the rich symplectic-topological rigidity near an anti-canonical divisor in a Fano manifold. We show that Kähler-Ricci flow on a Fano manifold decomposes into two purely symplectic processes: symplectic deflation along a (pluri)anticanonical divisor and an exact symplectic deformation in the divisor complement $X - D$. We view Theorem 1.1 result as a natural toric realization of this decomposition. In light of [SSW13] and others, we discuss how ψ_t is a C^2-obstruction to the optimal bound for the decay rate of the fiber under ω_t and urge a better understanding of its regularity. We present our work as a completely new approach to the study of Kähler-Ricci flow.

2. Kähler Ricci Flow Relative to a Divisor

2.1. Symplectic Deflation Near Plurianticanonical Divisors.

The cohomological solution

$$[\omega_t] = [\omega_0] - tc_1(X)$$

to the Kähler-Ricci flow implies that the flow simply deforms the cohomology class of the Kähler form linearly along the first Chern class. Let (X, g, ω) be a Fano (or Calabi-Yau) surface. We say a divisor $D \subset X$ is *plurianticanonical* if it is Poincare dual to a positive multiple of the first Chern class $c_1(X)$, write

$$PD[D] = kc_1(X).$$

Equivalently, D finds itself in the same divisor class as the k-plurianticanonical bundle $D \in |-K_X^{\otimes k}|$. For now, we will assume that this divisor is smooth. The normal bundle

$$\nu(D) \to \Sigma$$

provides a symplectic model for a symplectic tubular neighborhood $N_D \subset X$. The Thom class of this bundle may be represented by a 2-form $\omega_D \in \Omega^2_{dR}(N_D) \hookrightarrow \Omega^2_{dR}(M)$ compactly supported near the divisor. Consider the family of de Rham forms

$$\omega_t = \omega - t\omega_D$$

for $t \in [0, \varepsilon]$. This form is symplectic for $\varepsilon > 0$ sufficiently small. The 1-parameter family (X, ω_t) is a symplectic deformation known as an *symplectic deflation along* D. The class $c_1(-K_X^{\otimes k})$ restricts to the Thom class of the divisor D on any tubular neighborhood of D. Via the representative ω_D of the Thom class, the cohomological solution becomes

$$[\omega_t] = [\omega_0] - \frac{t}{k}[\omega_D].$$

Cohomologically, we see that Kähler-Ricci flow appears to be simply performing symplectic deflation along D.

2.2. **Volume evolution and finite collapse.** If $\Sigma \subset M$ is a 2-cycle, such as a J-holomorphic curve or some other divisor, then along the deformation we have

$$\begin{aligned} Vol(\Sigma, \omega_t) &= \langle [\omega_t], [\Sigma] \rangle \\ &= \int_\Sigma \omega_t \\ &= \int_\Sigma \omega - \frac{t}{k}\int_\Sigma \omega_D \\ &= Vol(\Sigma, \omega) - \frac{t}{k}[\Sigma] \cdot [D] \end{aligned}$$

so

$$\frac{d}{dt}Vol(\Sigma, \omega_t) = -\frac{[\Sigma] \cdot [D]}{k}$$

and thus the change of the area of a 2-cycle Σ along the deflation depends on the sign of its intersection with the divisor D and its speed depends on the number of intersection points. This also applies to the divisor D itself

$$\frac{d}{dt}Vol(D, \omega_t) = -\frac{[D] \cdot [D]}{k}$$

where $\ell = [D] \cdot [D]$ is the self-intersection number of the divisor (assumed to be a smooth surface). For this reason, positive divisors induce an upper bound on the existence time of each curve under Kähler-Ricci flow.

$$T_\Sigma = \frac{kVol(\Sigma, \omega)}{[\Sigma] \cdot [D]}$$

We let T_{max} be the maximum ε along which $[\omega_t]$ remains positive along the deflation. Then we may bound this time by the value T_D when $Vol(D, \omega_{T_D}) = 0$ so

$$T_{max} \leq T_D = \frac{kVol(D, \omega)}{[D] \cdot [D]}.$$

Suppose now that D has simple normal crossing singularities. One may still perform deflation just as easily. Suppose that a simple normal crossing (snc)-divisor D decomposes into smooth components as

$$D = \bigcup_{i=1}^{N} D_i.$$

We let ω_i be a compactly supported 2-form representing the the Thom class associated to the normal bundle $\nu(D_i) \to D_i$. Then letting $\omega_D = \sum_{i=1}^{N} \omega_i$, we may form $\omega_t = \omega - \frac{t}{k}\omega_D$ as before. Now we have

$$\frac{d}{dt} Vol(D_i, \omega_t) = -\frac{[D_i] \cdot [D]}{k} = -\frac{d_i + k_i}{k}$$

where d_i is the *degree* of the component $D_i \subset D$ in its divisor graph Γ_D and $k_i = [D_i] \cdot [D_i]$. Depending on the sign of this integer, the volume of a component may increase or decrease. If $d_i + k_i$ is positive, then the component collapses at time

$$T_{D_i} = \frac{kVol(D_i, \omega)}{d_i + k_i}$$

and we may refine our estimate

$$T_{max} \leq \min_i T_{D_i} \leq T_D.$$

2.3. **Deflation plus an exact deformation in the complement.** As we have seen, if $D \subset (X, g, \omega)$ is *smooth* and plurianticanonical, we may choose the 2-form ω_D representing its Thom class to be supported in a symplectic tubular neighborhood N_D of D modeled on the normal bundle of D. Because of the cohomological identity

$$[\omega_t] = [\omega] - \frac{t}{k}[\omega_D]$$

then we must have

$$\int_\Sigma \omega_t - (\omega - \frac{t}{k}\omega_D) = 0$$

for every 2-cycle Σ, and so $\omega_t - (\omega - \frac{t}{k}\omega_D)$ is an exact 2-form. This allows us to write

$$\omega_t = \omega - \frac{t}{k}\omega_D + d\beta_t$$

for some 1-form $\beta_t \in \Omega^1_{dR}(X)$. Thus we see that Kähler-Ricci flow differs from actual symplectic deflation only by the exact form $d\beta_t$.

Since $Supp(\omega_D) \subset N_D$, in the complement $\overline{X} = X - N_D$, we have $\omega_D = 0$ and the deformation becomes

$$\omega_t = \omega + d\beta_t$$

so Kähler-Ricci flow is a purely exact symplectic deformation there.

Remark 2.1. One can generalize this setup to N_D a regular symplectic neighbohood of a divisor with simple normal crossings. These neighborhoods are built from smoothings of plumbings of tubular neighborhoods of their smooth components. See for example [LM24].

These straightforward observations allow one to conclude

Theorem 2.2. *Let (X, g, ω) be a Fano variety and let $D \subset X$ be a smooth (or snc) k-plurianticanonical divisor. Then under Kähler-Ricci flow (X, g_t, ω_t) decomposes into two simultaneous processes:*

$$\omega_t = \omega_t^{Deflation} + \omega_t^{Exact}$$

(1) ***Symplectic deflation*** *along D supported in a tubular neighborhood $N_D \subset X$ with*

$$\omega_t^{Deflation} = \omega - \frac{t}{k}\omega_D$$

and

(2) *a global* ***exact symplectic deformation***

$$\omega_t^{Exact} = d\beta_t.$$

This decomposition presents a global organizing framework for Kähler-Ricci flow in the Fano setting. This perspective can be further refined in case of toric surfaces and its consideration there will lead to our proof of Theorem 1.1.

3. Toric Varieties

We now specialize to a smooth proper toric surface (X, g, ω). Such a manifold admits a *toric moment map* $\mu\colon X \to \mathbb{R}^2$. The image of the moment map is a closed convex polytope $P \subset \mathbb{R}^2$, its *Delzant polytope* or *moment polytope*. The preimage of the interior of the polytope $U = \mu^{-1}(P - \partial P)$ is an open dense submanifold with a free $\mathbb{T}^2$-action. The fiber $T_y = \mu^{-1}(y)$ over an interior point $y \in P$ is a torus which is invariant under the Hamiltonian $\mathbb{T}^2$-action.

3.1. Convex Toric Potentials.

A convex polytope can be described via a collection of linear functions $\{\ell_k\}_{k=1}^N$ defining its faces: Each face $F_k \subset \partial P$ of the polytope lies on a unique line which has a unique outward-pointing primitive normal vector $\vec{n}_k$. The Delzant polytope P may be presented as an intersection of the half spaces defined by inequalities (c.f. [Gui94])

$$\begin{gathered}\langle y, \vec{n}_1\rangle \geq \lambda_1\\ \langle y, \vec{n}_2\rangle \geq \lambda_2\\ \vdots\\ \langle y, \vec{n}_N\rangle \geq \lambda_N\end{gathered}$$

with $\lambda_k \in \mathbb{Z}$. We may then define the functions

$$\ell_k(y) = \langle y, \vec{n}_k\rangle - \lambda_k$$

so that a point y is in the interior of P if and only if $\ell_k(y) > 0$ for all k.

Following the framework of Abreu and Guillemin [Abr98][Gui94], each Kähler potential $\phi\colon X \to \mathbb{R}$ corresponds under the Lengendre transform to a convex function $u\colon U \to \mathbb{R}$ over the interior of the moment polytope P. In toric action-angle coordinates $(y_1, y_2, \theta_1, \theta_2)$, there is a canonical choice of u (c.f. [Abr98][Gui94])

$$u(y) = \frac{1}{2}\sum_{k=1}^{N} \ell_k(y)\log(\ell_k(y)).$$

By setting $x_i = \frac{\partial u}{\partial y_i}$, we may obtain a Kähler potential ϕ for a Kähler form ω in (x_1, x_2)-coordinates via Legendre transform. We may exponentiate these to toric coordinates $z_i = e^{x_i + i\theta_i}$ for the interior $X - D$. We also have

$$\frac{\partial \phi}{\partial x_i} = y_i.$$

$$[\omega] = \sum_{k=1}^{N} \lambda_k [\omega_{D_k}]$$

and

$$c_1(X) = \sum_{k=1}^{N} [\omega_{D_k}]$$

A quick computation shows

Proposition 3.0.1. *We have*

$$du = \frac{1}{2} \sum_{k=1}^{N} [1 + \log(\ell_k(y))] d\ell_k$$

$$\nabla^2 u = \frac{1}{2} \sum_{k=1}^{N} \frac{d\ell_k \otimes d\ell_k}{\ell_k(y)}$$

for u, ℓ_k defined above.

3.2. **Toric Ruled Surfaces.** We say a holomorphic submersion $f \colon X \to \Sigma$ is *compatible* with a toric structure if its restriction to each fiber is toric. For toric surfaces, this implies that each fiber is a toric Riemann surface which then implies that it is diffeomorphic to S^2. Thus $f \colon X \to \Sigma$ is actually a $\mathbb{P}^1$-bundle over Σ. This is sometimes referred to as a T^2-invariant ruling on X. We shall refer to X as a *toric ruled surface*.

Example 3.1. Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$ with standard toric moment map $\mu \colon X \to \mathbb{R}^2$ given by the product of the standard moment map for $\mathbb{P}^1$. The image of μ is a square $P = [0,1]^2 \subset \mathbb{R}^2$. The holomorphic submersion $f \colon X \to \mathbb{P}^1$ given by $(p_1, p_2) \mapsto p_1$ is compatible with the toric structure.

Example 3.2. Take $\mathbb{CP}^2$ together with an embedded projective line $L \subset \mathbb{CP}^2$. We may fit this line to a Lefschetz pencil on $\mathbb{CP}^2$ whose base locus is a single point p. Blowing up this point p, we obtain a Lefschetz fibration $F \colon X \to \mathbb{P}^1$. The toric structure on $\mathbb{CP}^2$ may be used to construct a toric structure on X. Via a transformation of $\mathbb{CP}^2$, we can ensure that our Lefschetz fibration is also compatible with this toric structure.

Example 3.3. More generally, we consider a Hirzebruch surface $X_{b-a} = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b))$. This carries an effective $\mathbb{T}^2$-action which descends from the effective $\mathbb{T}^2$-action on the rank-2 bundle $\mathcal{O}(a) \oplus \mathcal{O}(b)$.

Possessing a compatible Hamiltonian S^1 action on fibers leads to remarkable rigidity. For each fixed fiber, such an action represents a subgroup $S^1 \hookrightarrow \mathbb{T}^2$, a periodic 1-parameter family of elements. Such a family is characterized by its slope at the identity. This is represented by a vector $v \in \mathfrak{t}$ coming from some parameterization of S^1. In the standard parameterizaiton of $\mathbb{T}^2$, this is a vector of some fixed rational slope. Such a holomorphic submersion thus represents a continuous family of rational slopes which only exists if the family is locally constant. It then follows that such an action may be represented by a non-vanishing assigment of parallel representing vectors $\sigma \colon \Sigma \to \mathfrak{t}$. By definition, each of these generates a vector field $X_{\sigma(q)}$ which is tangent to its corresponding fiber $f^{-1}(q)$. From such a vector, one can choose a single vector $v \in \mathfrak{t}$ which generates a vector field X_v tangent to all fibers simultaneously (in fact every vector in the image of σ satisfies this). This yields a potential condition to check for establishing compatibility for more specific examples.

Proposition 3.3.1. *A submersion $f \colon X \to \Sigma$ is compatible with the toric structure on X if and only if there is a primitive vector $v \in \mathfrak{t}$ such that, for each fiber F, $X_v \in TF$ everywhere.*

3.3. **A shrinking family of toric metrics.** From the presentation of the moment polytope P via the collection of inequalities

$$\langle y, n_k \rangle \geq \lambda_k$$

we have a canonical toric divisor: Let $P_k \subset P$ denote the face of P defined by $\ell_k = 0$. Then $D_k = \mu^{-1}(P_k)$ is a smooth sphere and

$$D = \bigcup_{k=1}^{N} D_k$$

is a normal crossing divisor. The cohomology class of this divisor depends on n_k but not on λ_k. We consider the family P_t of polytopes defined by

$$\langle y, n_k \rangle \geq \lambda_k - t$$

for $t \in [0, T]$. This canonically defines a 1-parameter family of toric Kähler metrics $\widetilde{\omega}_t$ via the Guillemin potentials

$$\widetilde{u}_t = \frac{1}{2} \sum_{k=1}^{N} \ell_{k,t}(y) \log(\ell_{k,t}(y))$$

We may present $[\omega]$ and $c_1(X)$ via the toric divisors [Gui94], we have

$$[\widetilde{\omega}_t] = \sum_{k=1}^{N} 2\pi(\lambda_k - t)[\omega_{D_k}]$$

and

$$c_1(X) = \sum_{k=1}^{N} 2\pi[\omega_{D_k}]$$

after choosing compactly supported 2-forms ω_{D_k} as before. This allows us to write

$$[\widetilde{\omega}_t] = [\omega] - tc_1(X).$$

Thus our family follows the cohomological profile of a Kähler-Ricci flow starting from ω.

3.4. **The geometry of the fiber.** The moment map $\mu\colon X \to P$ restricts to a fiber F to yield a map $\mu|_F : F \to P$. Its image is a curve $\gamma_F \subset P$ with endpoints on ∂P. Let $v \in \mathfrak{t}$ be a vector generating a compatible Hamiltonian S^1-action on F. We can extend v to a basis $\{v, w\}$ for the Lie algebra $\mathfrak{t}$ which we may dualize to obtain a pair of moment coordinates (y_1, y_2) for $\mathfrak{t}^*$. This gives rise to a pair of toric coordinates (z_1, z_2) in $X - D$.

The moment map $\mu_F\colon F \to \mathbb{R}$ for its compatible Hamiltonian S^1-action is $\mu_F = \mu_1$. Its image is the interval $I_F \subset \mathbb{R}$ given by $I_F = [s_-, s_+]$. Using this to fix action angle coordinates (s, φ), we may write the fiber metric $g_F = g|_F$ in the form

$$g_F = S\, ds^2 + S^{-1}\, d\varphi^2$$

where $S = \Delta u_F$ for some convex toric potential $u_F\colon I_F \to \mathbb{R}$.

We will compute Δu_F from $\nabla^2 u$. Recall that for the Guillemin potential, we have

$$\nabla^2 u = \frac{1}{2} \sum_{k=1}^{N} \frac{d\ell_k \otimes d\ell_k}{\ell_k(y)}$$

We may use the moment coordinate s to form a parameterization $\gamma_F(s)$ of the μ-moment image of F. It follows that

$$\nabla^2 u|_F = \frac{1}{2} \sum_{k=1}^{N} \frac{d\ell_k(\dot{\gamma}_F(s))^2}{\ell_k(\gamma_F(s))}.$$

By letting $f_k(s) = \ell_k(\gamma_F(s))$, we may write this more compactly as

$$\nabla^2 u|_F = \frac{1}{2} \sum_{k=1}^{N} \frac{(f_k')^2}{f_k}$$

Since the fiber is given on the open torus orbit by $z_2 = c$, we have

$$x_2 = \frac{\partial u}{\partial y_2} = \log |c|$$

along F. Writing

$$\alpha_k = -\frac{\partial \ell_k}{\partial y_2},$$

we therefore obtain

$$\frac{\partial u}{\partial y_2} = -\frac{1}{2} \sum_{k=1}^{N} \alpha_k (1 + \log \ell_k) = \log |c|$$

which exponentiates to

$$\prod_{k=1}^{N} \ell_k^{\alpha_k} = C$$

We let $\mathcal{A}_+ = \{\alpha_k : \alpha_k \geq 0\}$ and $\mathcal{A}_- = \{\alpha_k : \alpha_k < 0\}$, then

$$\prod_{k=1}^{N} \ell_k^{\alpha_k} - C = 0$$

is equivalent to

$$\prod_{\alpha_+\in\mathcal{A}_+}\ell_k^{\alpha_+} - C\prod_{\alpha_-\in\mathcal{A}_-}\ell_k^{-\alpha_-} = 0.$$

The left hand side is polynomial in (y_1, y_2) and the moment curve γ_F is the vanishing locus of this polynomial intersected with P.

4. Kahler Ricci Flow on Toric Ruled Surfaces

4.1. **Collapsing and non-collapsing.** We let

$$\omega_\Sigma = \frac{f_*(\omega\wedge\omega)}{Vol(F,\omega)}$$

where $f_*\colon \Omega^4_{dR}(X)\to\Omega^2_{dR}(\Sigma)$ denotes fiber integration. We have

$$Vol(X,\omega) = \int_X \omega\wedge\omega = \int_\Sigma f_*(\omega\wedge\omega)$$

and so $Vol(X,\omega) = Vol(F,\omega)Vol(\Sigma,\omega_\Sigma)$.

Under Kähler-Ricci flow, we have

$$\begin{aligned} Vol(X,\omega_t) &= \int_X \omega_t\wedge\omega_t \\ &= \int_X (\omega - tc_1(X))\wedge(\omega - tc_1(X)) \\ &= \int_X \omega\wedge\omega - 2t\int_X c_1(X)\wedge\omega + t^2\int_X c_1(X)\wedge c_1(X) \\ &= Vol(X,\omega) - 2tVol(D,\omega) + t^2[D]\cdot[D] \end{aligned}$$

and of course

$$Vol(F,\omega_t) = Vol(F,\omega) - t[F]\cdot[D]$$

A basic understanding of collapsing behavior of X under Kähler-Ricci flow comes first from understanding the discriminant of $Vol(X,\omega_t)$ which is quadratic in t. If

$$Vol(X,\omega) > \frac{Vol(D,\omega)^2}{[D]\cdot[D]}$$

then total volume remains non-collapsed for all $t < T$ which means so does the volume of F. Kähler-Ricci flow thus continues until some $t = T$ with

$$T \le T_F$$

and

$$T \le T_D.$$

If

$$Vol(X,\omega) \le \frac{Vol(D,\omega)^2}{[D]\cdot[D]}$$

then X may collapse in several ways. We let $t = T_X$ denote the first positive root of $Vol(X, \omega_t)$. If strict inequality holds, then we have $T_X < T_D$ and so X collapses before D does. If $T_X < T_F$, then we have

$$\lim_{t \to T_X} Vol(\Sigma, \omega_\Sigma(t)) = 0$$

while the volume of the fiber remains positive. If $T_X = T_F$, then by L'Hôpital's rule

$$\lim_{t \to T_X} Vol(\Sigma, \omega_\Sigma(t)) = \lim_{t \to T_X} \frac{Vol(X, \omega_t)}{Vol(F, \omega_t)} = \lim_{t \to T_X} \frac{\partial_t Vol(X, \omega_t)}{\partial_t Vol(F, \omega_t)} = \lim_{t \to T_X} \frac{2Vol(D, \omega_t)}{[F] \cdot [D]}$$

and so $(\Sigma, \omega_\Sigma(t))$ remains non-collapsed unless $T_X = T_F = T_D$.

We will assume that

$$T = T_F < T_D$$

so that the Kähler-Ricci flow exists until volume of the fiber collapses and

$$\lim_{t \to T_X} Vol(\Sigma, \omega_t) > 0.$$

this is essentially the same setting as [SSW13] where it is required that

$$[\omega] - Tc_1(X) = [f^* \Omega]$$

for some Kähler metric Ω on Σ. Our collapsing condition is much weaker but we suspect the stronger condition should also hold in this setting as well.

4.2. **Diameter decay under Kähler-Ricci flow.** We are now ready to prove our main theorem. The diameter of F is defined to be the supremum over all pairs of points $p, q \in F$ of g_F-distances $dist_{g_F}(p, q)$. Let p, q be a pair of points realizing the diameter. We may always join any two points in F by segments of a pair of meridians through the poles of F. This allows us to approximate the distance between p and q with curve with piecewise constant angle φ in action angle coordinates (s, φ). We thus may estimate the diameter to be

$$diam_{g_F}(F) \leq 2L[F]$$

Where $L[F]$ is the meridional length of F defined to be the length of any fixed meridian $m = \{\varphi = \varphi_0\}$ between the poles $p_- = \{s = s_-\}$ and $p_+ = \{s = s_+\}$. We have

$$L[F] = Length[m] = \int_{I_F} \sqrt{S}\, ds.$$

From our formula for S above,

$$S = \frac{1}{2} \sum_{k=1}^{N} \frac{(f_k')^2}{f_k}$$

this becomes the quantity

$$L[F] \leq \frac{1}{\sqrt{2}} \sum_{k=1}^{N} \int_{I_F} \frac{|f_k'|}{\sqrt{f_k}}\, ds.$$

Each integral on the right may be computed via the Jacobian formula which yields

$$\int_{I_F} \frac{|f_k'|}{\sqrt{f_k}}\, ds = \int_{\mathbb{R}} \sum_{s \in f_k^{-1}(t)} \frac{1}{\sqrt{t}}\, dt = \int_{f_{min}}^{f_{max}} \frac{N_k(t)}{\sqrt{t}}\, dt$$

where

$$N_k(t) = \#f_k^{-1}(t) = \#(\gamma_F \cap \{\ell_k = t\}).$$

The right hand side is finite and uniformly bounded above by a constant $K > 0$ by Bezout's theorem and so we have an upper bound

$$L[F] \leq \int_{f_{min}}^{f_{max}} \frac{K}{\sqrt{t}}\, dt = 2K(\sqrt{f_{max}} - \sqrt{f_{min}}) \leq 2K\sqrt{f_{max} - f_{min}}$$

It is straightforward to show $|f_k'| \leq D$ for some D. Thus we have

$$\sqrt{f_{max} - f_{min}} \leq D\sqrt{s_+ - s_-}$$

Since

$$Vol(F, \omega) = \int_F ds \wedge d\varphi = 2\pi(s_+ - s_-)$$

this yields the estimate

$$diam_{g_F}(F) \leq C\sqrt{Vol(F, \omega)}.$$

for some constant C. From this inequality follows our main theorem.

Theorem 1.1. *Let $\widetilde{\omega}_t$ be the canonical shrinking family of toric metrics starting at (X, g, ω). Then we have*

$$diam_{\widetilde{\omega}_t}(F) \leq C(T - t)^{1/2}$$

Proof. Along the family, we have

$$diam_{\widetilde{\omega}_t}(F) \leq C\sqrt{Vol(F, \widetilde{\omega}_t)}$$

Since $\widetilde{\omega}_t$ and ω_t are cohomologous,

$$Vol(F, \widetilde{\omega}_t) = Vol(F, \omega_t) = Vol(F, \omega) - t[F]\cdot[D] = [F]\cdot[D]\left(\frac{Vol(F, \omega)}{[F]\cdot[D]} - t\right)$$

and since

$$T = T_F = \frac{Vol(F, \omega)}{[F]\cdot[D]}$$

we have

$$Vol(F, \widetilde{\omega}_t) = [F]\cdot[D]\,(T - t)\,.$$

We may then estimate

$$diam_{\widetilde{\omega}_t}(F) \leq C\sqrt{[F]\cdot[D]}(T - t)^{1/2}$$

□

Along the actual Kähler Ricci flow, we have $\phi_t = \widetilde{\phi}_t + \psi_t$ so that

$$\omega_t = i\partial\overline{\partial}\phi_t = \widetilde{\omega}_t + i\partial\overline{\partial}\psi_t$$

The regularity of ψ_t is a direct obstruction to transferring this estimate to the Kähler-Ricci flow itself. In particular, uniform control of the second derivatives of ψ_t along the collapsing fibers would provide the natural bridge between the symplectic model and the sharp Kähler-Ricci flow estimate.

References

[Abr98] Miguel Abreu, *Kähler geometry of toric varieties and extremal metrics*, International Journal of Mathematics **9** (1998), no. 3, 641–651.

[Bed25] Alexander Bednarek, *Fano fibrations and twisted kähler–einstein metrics II: The kähler–ricci flow*, 2025.

[CT15] Tristan C. Collins and Valentino Tosatti, *Kähler currents and null loci*, Inventiones Mathematicae **202** (2015), no. 3, 1167–1198.

[FIK03] Mikhail Feldman, Tom Ilmanen, and Dan Knopf, *Rotationally symmetric shrinking and expanding gradient kähler–ricci solitons*, Journal of Differential Geometry **65** (2003), no. 2, 169–209.

[Fon14] Frederick Tsz-Ho Fong, *Kähler–ricci flow on projective bundles over kähler–einstein manifolds*, Transactions of the American Mathematical Society **366** (2014), no. 2, 563–589.

[Fon15] ______, *On the collapsing rate of the kähler–ricci flow with finite-time singularity*, Journal of Geometric Analysis **25** (2015), no. 2, 1098–1107.

[FZ17] Xin Fu and Shijin Zhang, *The kähler–ricci flow on fano bundles*, Mathematische Zeitschrift **286** (2017), no. 3–4, 1605–1626.

[Gui94] Victor Guillemin, *Kähler structures on toric varieties*, Journal of Differential Geometry **40** (1994), no. 2, 285–309.

[JS26a] Wangjian Jian and Jian Song, *Finite-time singularities of the kähler–ricci flow on fano bundles*, 2026.

[JS26b] ______, *Finite-time singularities of the kähler–ricci flow on fano bundles II*, 2026.

[LM24] Tian-Jun Li and Jie Min, *Local geometry of symplectic divisors with applications to contact torus bundles*, Proceedings of the International Consortium of Chinese Mathematicians **2019** (2024).

[She21] Xi Sisi Shen, *The kähler–ricci flow, holomorphic vector fields and fano bundles*, Transactions of the American Mathematical Society **374** (2021), no. 9, 6751–6768.

[Son14] Jian Song, *Finite-time extinction of the kähler–ricci flow*, Mathematical Research Letters **21** (2014), no. 6, 1435–1449.

[SSW13] Jian Song, Gábor Székelyhidi, and Ben Weinkove, *The kähler–ricci flow on projective bundles*, International Mathematics Research Notices **2013** (2013), no. 2, 243–257.

[ST08] Natasa Sesum and Gang Tian, *Bounding scalar curvature and diameter along the kähler ricci flow (after perelman)*, Journal of the Institute of Mathematics of Jussieu **7** (2008), no. 3, 575–587.

[SW11] Jian Song and Ben Weinkove, *The kähler–ricci flow on hirzebruch surfaces*, Journal für die reine und angewandte Mathematik **659** (2011), 141–168.

[SW13] ______, *Contracting exceptional divisors by the kähler–ricci flow*, Duke Mathematical Journal **162** (2013), no. 2, 367–415.

[SW14] ______, *Contracting exceptional divisors by the kähler–ricci flow II*, Proceedings of the London Mathematical Society **108** (2014), no. 6, 1529–1561.

[Tia08] Gang Tian, *New results and problems on kähler–ricci flow*, Géométrie différentielle, physique mathématique, mathématiques et société (ii), 2008, pp. 71–92.

[TZ06] Gang Tian and Zhou Zhang, *On the kähler–ricci flow on projective manifolds of general type*, Chinese Annals of Mathematics, Series B **27** (2006), no. 2, 179–192.

[TZ18] Valentino Tosatti and Yuguang Zhang, *Finite time collapsing of the kähler–ricci flow on threefolds*, Annali della Scuola Normale Superiore di Pisa **18** (2018), no. 1, 105–118.

[XZ26] Tongxin Xu and Zhenlei Zhang, *Finite time singularities of collapsing kähler ricci flow on ruled surfaces*, 2026.

[ZZ23] Lei Zhang and Zhenlei Zhang, *On the finite time collapsing rate of kähler–ricci flow on projective bundles*, Proceedings of the American Mathematical Society **151** (2023), no. 12, 5385–5389.

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