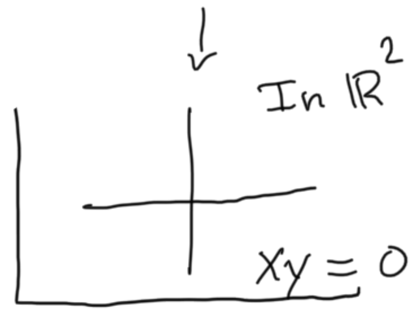
Motivation:

What are the boundaries of $\mathcal{O}p(D)$ of D; Symplectic. Normal Crossing divisors ?

Always working in dimension 4

Model Normal Crossing:

$$X = \{ z_1 z_2 = 0 \} \subseteq \mathbb{C}^2$$



- X decomposes into $X = \{z_1 = 0\} \cup \{z_2 = 0\}$
- X has a single isolated singular point at $(0,0)$.
- $\{z_1 = 0\} \pitchfork \{z_2 = 0\}$

Symplectic divisors with normal crossings:

A subspace $D \subseteq (M^4, \omega)$ is a symplectic NC-Divisor if:

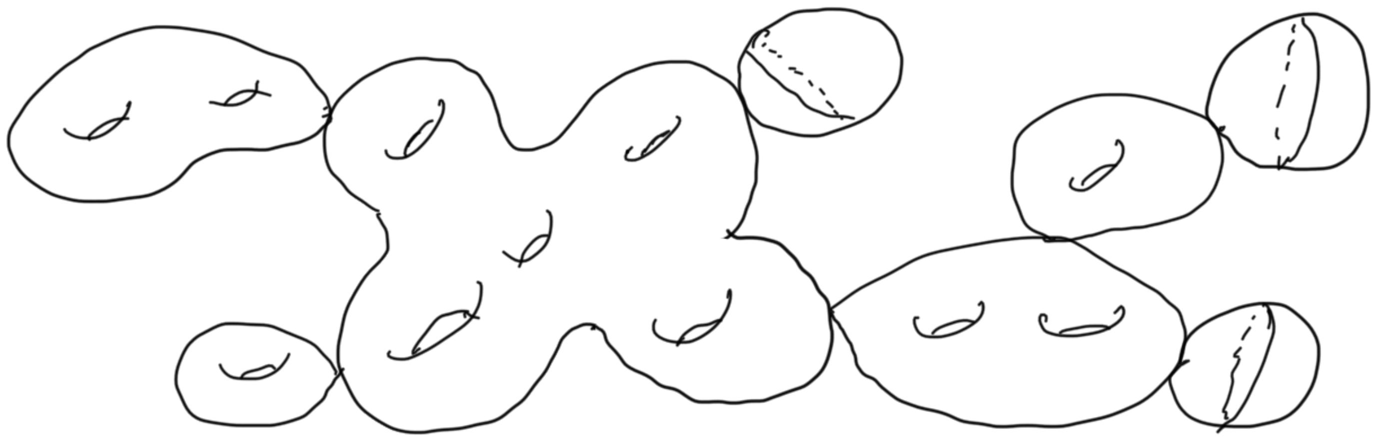
(1) $D = \bigcup S_i$ $\qquad$ S_i = Smooth Symplectic Surface

(2) Only intersect in pairs

at a single point.

(3) Intersections locally are model normal crossings

Topologically: $D \approx \bigvee S_i$



Q: What is a good model for $\mathcal{O}p(D)$?

↳ Related: What is a good model for $\mathcal{O}p(\Sigma)$ where Σ is a symplectic surface?

A:

- $\mathcal{O}p(\Sigma)$ can be modeled as the normal bundle of Σ.

$$\begin{array}{c} \nu(\Sigma) \\ \downarrow \\ \Sigma \end{array}$$

- $\nu(\Sigma)$ is a (symplectic) $\mathbb{D}^2$-bundle

D^2-bundles:

- To every $\mathbb{D}^2$-bundle E we may associate a vector bundle $V_E \longrightarrow \Sigma$

- Characteristic classes of V_E are invariants of E
- $e(V_E) \in H^2(\Sigma;\mathbb{Z}) \cong \mathbb{Z}$
- $e(E) := e(V_E)$ completely determines $\nu(\Sigma)$

For symplectic surfaces in symplectic 4-manifolds:

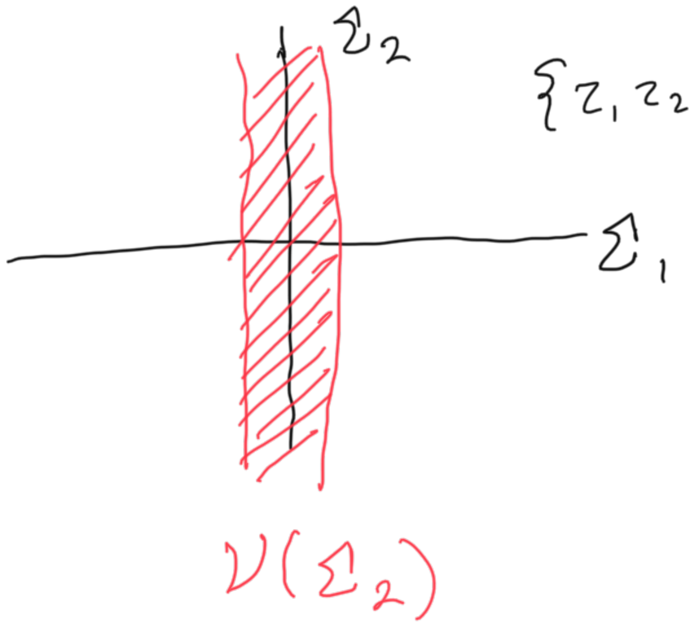
$$e(\nu(\Sigma)) = \Sigma \cdot \Sigma$$

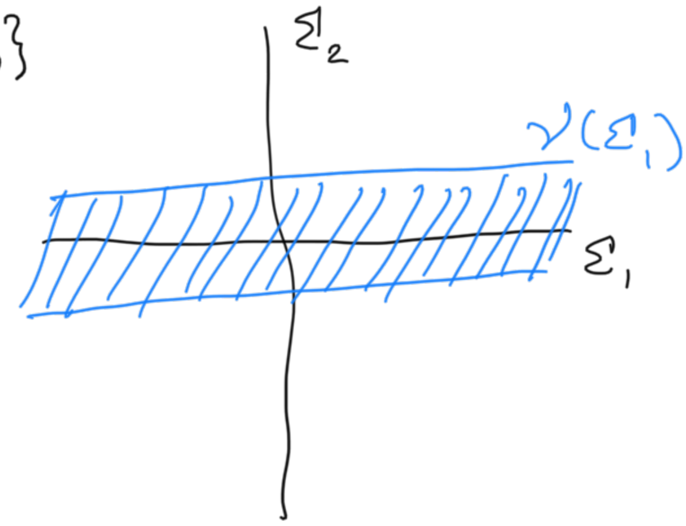
Away from normal crossing $\nu(\Sigma)$ works.

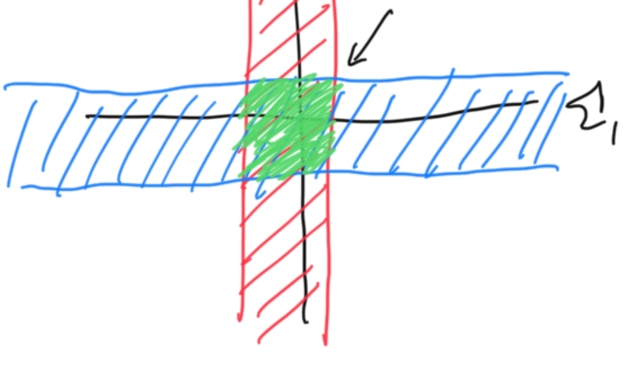
What about <u>near the normal crossing</u>?

In $\mathbb{R}$

$\{z_1 z_2 = 0\}$







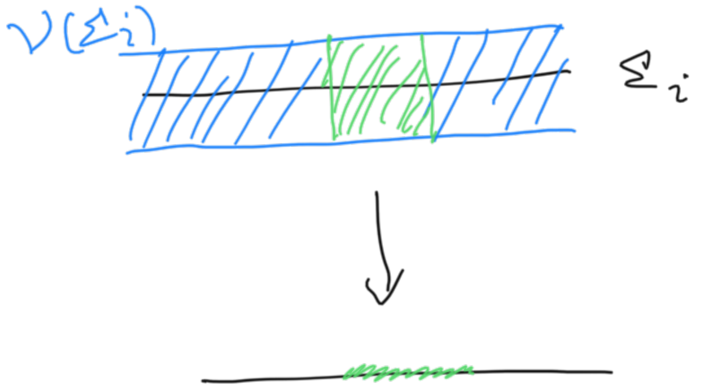
$\boxed{\;= \mathbb{D}^2 \times \mathbb{D}^2\;}$

Polydisk.

$Op(\Sigma_1 \cup \Sigma_2)$ can be modeled as a <u>plumbing</u> of $\nu(\Sigma_1) + \nu(\Sigma_2)$

<u>Plumbing</u>:

(1) For each Σ_i fix $\Delta_i \subseteq \nu(\Sigma_i)$ diffeomorphic to $\underline{\mathbb{D}^2 \times \mathbb{D}^2}$



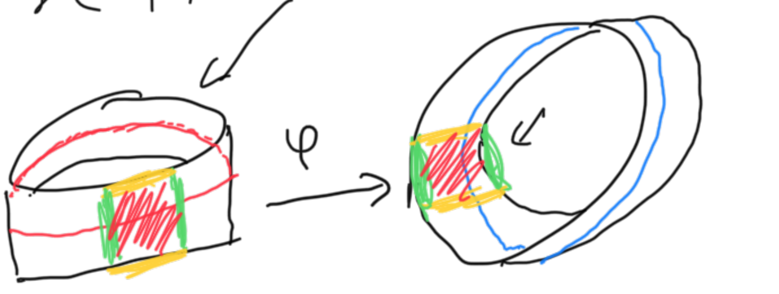
 $= \Delta_i$

$$\begin{array}{c} \Delta_i \\ \downarrow \\ \Sigma_i \end{array}$$ is a trivial $\mathbb{D}^2$-bundle

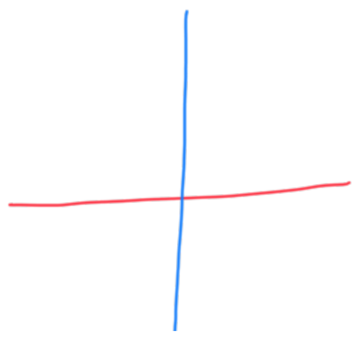
(2) Glue along the map

$$\varphi : \Delta_1 \longrightarrow \Delta_2$$

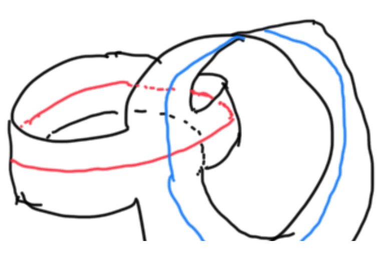
In coordinates: $\varphi(z_1, z_2) = (z_2, z_1)$

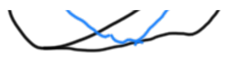


Introduces $\pitchfork$

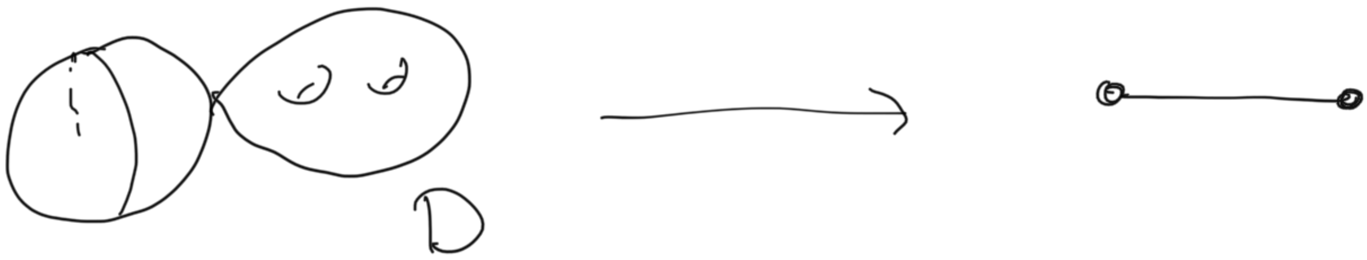


$\nu(\Sigma_1) \natural \nu(\Sigma_2)$





To model $Op(D)$, plumb disk bundles $\nu(S_i)$ according to their connectivity graph Γ

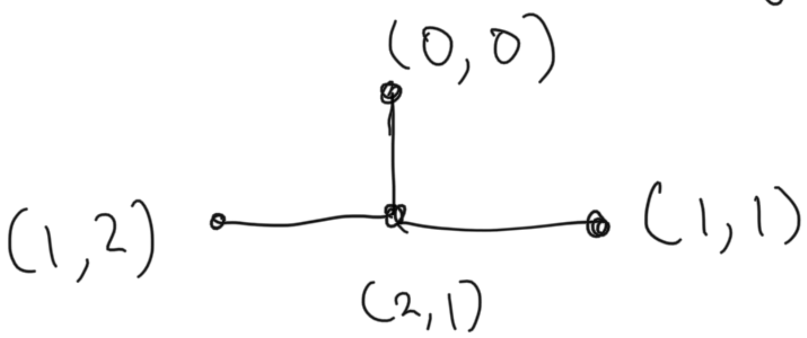


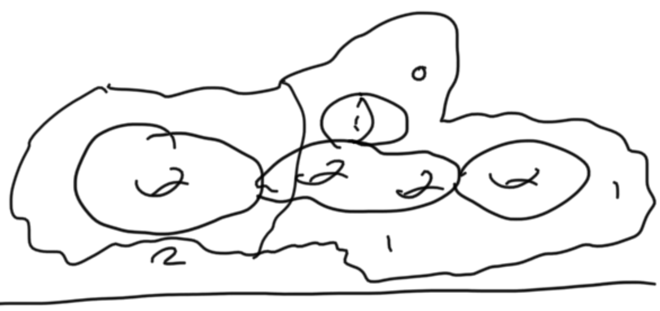
Convenient to decorate Γ

g_i = genus

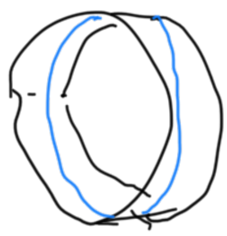
v_i = self intersection.

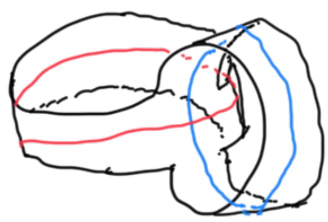
Model all possible neighborhoods by decorated graphs.





Q: What about $\partial Op(D)$? These are S^1-bundles!





$= \partial(E_1 \& E_2)$

$\partial(E_1 \cup E_2) =$ 

Plumbing surgery: let Δ_i be as above $\mathbb{D}^2 \times \mathbb{D}^2$

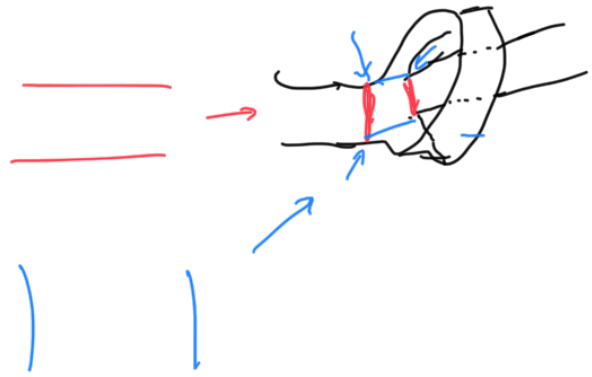
let $\partial_V \Delta_i \approx \mathbb{D}^2 \times S^1$

$\partial_H \Delta_i \approx S^1 \times \mathbb{D}^2$

$\partial(\Delta_i) = \partial_V \Delta_i \cup \partial_H \Delta_i$

$\partial_V \Delta_1 \xrightarrow{\text{glue}} \partial_H \Delta_2$

$\partial_H \Delta_1 \xrightarrow{\text{glue}} \partial_H \Delta_2$



$$\mathrm{Int}(\partial_{V/H} \Delta_i) \subseteq \mathrm{Int}\left(\nu(\Sigma_1) \,\S\, \nu(\Sigma_2)\right)$$

Corners: $\partial_V \Delta_i \cap \partial_H \Delta_i \approx S^1 \times S^1$

Glued along the map

$$\varphi : \mathrm{Corner}(\Delta_1) \longrightarrow \mathrm{Corner}(\Delta_2)$$

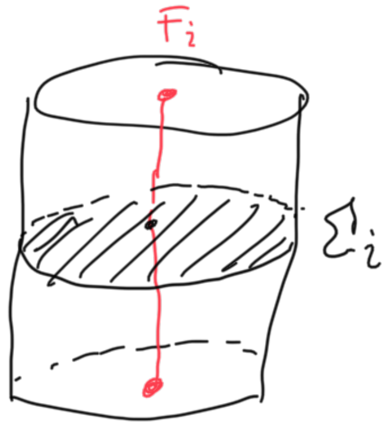
In coordinates $(\theta_1, \theta_2) \longmapsto (\theta_2, \theta_1)$

∂ Surgery:

(1) Remove $\partial_V \Delta_i \subseteq \partial \nu(\Sigma_i)$

"Delete a tubular neighborhood of a fixed fiber"

i.e. Remove a solid torus.



(2) Glue along common boundy torus.

$$\varphi_0 \in \mathrm{Aut}(T^2) \simeq S\bar{L}(2,\mathbb{Z})$$

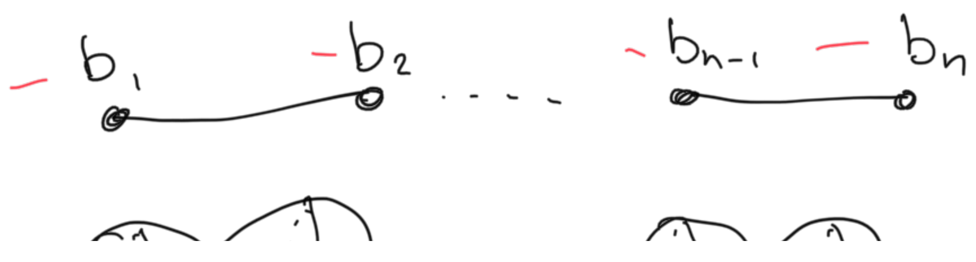
$$\varphi_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \det \varphi_0 = -1$$

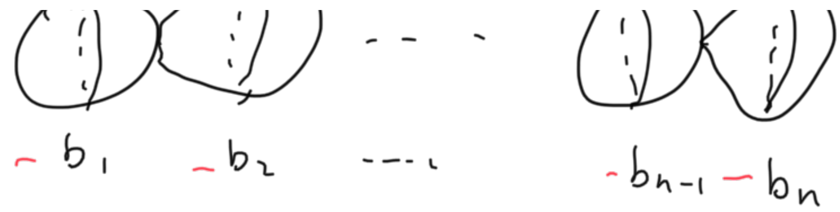
Is how we glue after fixing a collar neighborhood.

<u>Details</u>: How do self intersections factor in?

<u>A</u>: Exercise. (Hint: Collar neighborhood)

<u>Theorem</u>: Let Y_Γ be the ∂ of result of plombing according to





Then $Y_\Gamma \cong L(p,q)$

when $$\frac{p}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cdots - \cfrac{1}{b_{n-1} - \cfrac{1}{b_n}}}}$$

Proof Sketch:

Lens Space: $p, q \in \mathbb{Z}$ relatively prime

$$L(p,q) := S^3 \subseteq \mathbb{C}^2 \big/ \sim$$

$$(e^{i\theta_1}, e^{i\theta_2}) \longmapsto \left(e^{i\theta_1/q}, e^{i\frac{p\theta_2}{q}}\right)$$

Standard Decomposition of S^3

$$S^1 \times \mathbb{D}^2 \cup_{\phi_{p,q}} S^1 \times \mathbb{D}^2$$

where $\phi_{p,q} \in SL(2,\mathbb{Z})$ is of the form

$$\begin{bmatrix} r & s \\ p & q \end{bmatrix}$$

S^1-bundles over S^2

$E \to S^2$ decomposes as

$$E = D_- \times S^1 \cup_\psi D_+ \times S^1$$

$\psi \in \tilde{SL}(2,\mathbb{Z})$ is the clutching function

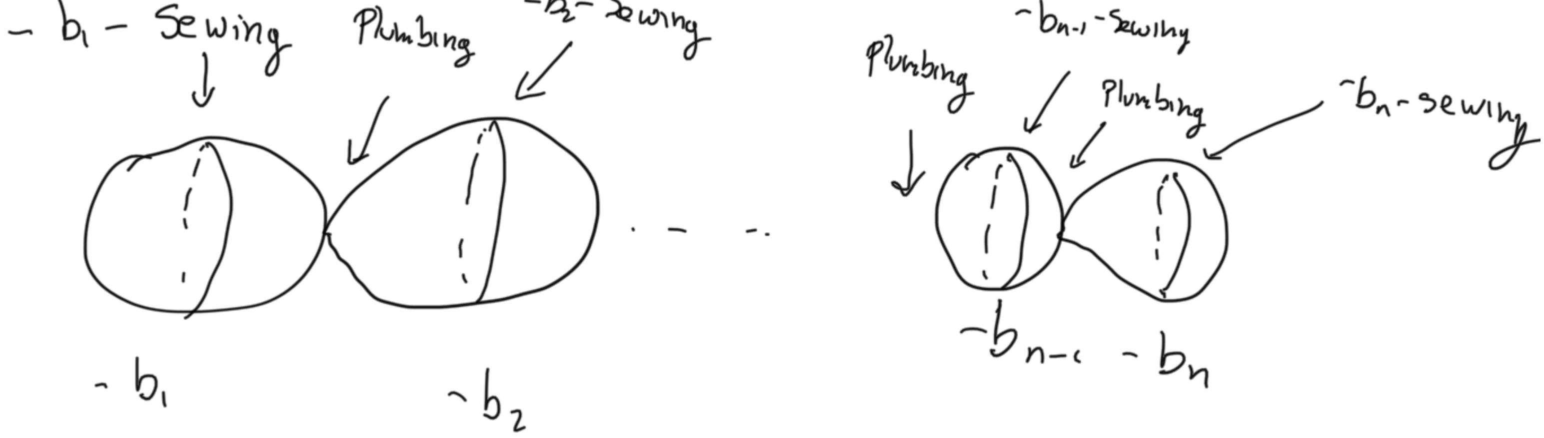
D_-, D_+ are hemispheres of S^2.

$$\psi = \begin{bmatrix} -1 & 0 \\ -e(E) & 1 \end{bmatrix} \quad (\det \psi = -1)$$

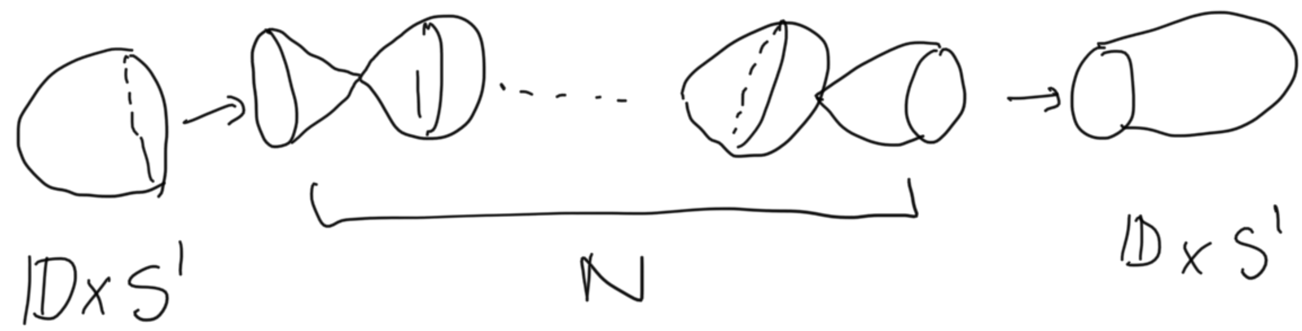
Forces $[\partial D_-^2 \times \{1\}] = e(E)[S^1] + [\partial D_+]$ in $H_1(E)$.

This operation is called Sewing.

A linear plumbing is given by a sequence of Sewings + plumbings



Cut along <u>first</u> and <u>last</u> Sewing sites



up to corners

$$N \underset{\text{homeo}}{\approx} S^1 \times [0,1] \times S^1 \approx T^2 \times [0,1]$$

Choose a diffeo of N so that the first diffeo on left is $\begin{bmatrix} -1 & 0 \\ b_1 & 1 \end{bmatrix}$

All we need is the Diffeo on the right.

<u>Full diffeo:</u>

$$\begin{bmatrix} -1 & 0 \\ b_n & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdots \overset{\text{sew}}{\begin{bmatrix} -1 & 0 \\ b_1 & 1 \end{bmatrix}} \cdot \overset{\text{plumb}}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \cdot \overset{\text{sew}}{\begin{bmatrix} -1 & 0 \\ b_1 & 1 \end{bmatrix}} = \begin{bmatrix} r & s \\ p & q \end{bmatrix}$$

let $\quad p_0 = 1 \qquad q_0 = 0$

$\qquad\; p_1 = b_1 \qquad q_1 = 1$ $\quad$ so $\begin{bmatrix} -p_0 & -q_0 \\ p_1 & q_1 \end{bmatrix}$

First 3 multiplications

$$\begin{bmatrix} 0 & 1 \\ \end{bmatrix}\begin{bmatrix} -p_0 & -q_0 \\ \end{bmatrix} = \begin{bmatrix} p_1 & q_1 \\ \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 & q_1 \end{bmatrix} = \begin{bmatrix} -p_0 & -q_0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ b_2 & 1 \end{bmatrix} \begin{bmatrix} p_1 & q_1 \\ -p_0 & -q_0 \end{bmatrix} = \begin{bmatrix} -p_1 & -q_1 \\ \underline{b_2 p_1 - p_0} & \underline{b_2 q_1 - q_0} \end{bmatrix}$$

Set $p_2 = b_2 p_1 - p_0$ + $q_2 = b_2 q_1 - q_0$

$$\begin{bmatrix} -p_1 & -q_1 \\ p_2 & q_2 \end{bmatrix}$$

<u>In general</u>: $p_k = b_k p_{k-1} - p_{k-2}$

$q_k = b_k q_{k-1} - p_{k-2}$

$p_0 = 1 \qquad q_0 = 0$

$p_1 = b_1 \qquad q_1 = 1$

$$\begin{bmatrix} r & s \\ p_n & q_n \end{bmatrix}$$

Need

$$\frac{p_n}{q_n} = \frac{b_n p_{n-1} - p_{n-2}}{b_n q_{n-1} - q_{n-2}} \qquad \left(\text{factor out } b_n \right)$$

$$= \frac{p_{n-1} - \frac{p_{n-2}}{b_n}}{q_{n-1} - \frac{q_{n-2}}{b_n}} \qquad \left(\text{expand } p_{n-1}, q_{n-1} \right)$$

$$= b_{n-1} p_{n-2} - p_{n-3} - \frac{p_{n-2}}{b_n}$$

$$b_{n-1} q_{n-2} - q_{n-3} - \frac{q_{n-2}}{b_n}$$

$$= \frac{\left(b_{n-1} - \frac{1}{b_n}\right) p_{n-2} - p_{n-3}}{\left(b_{n-1} - \frac{1}{b_n}\right) q_{n-2} - q_{n-3}}$$

Group p_{n-2} q_{n-2}

$$= \frac{p_{n-2} - \dfrac{p_{n-3}}{\left(b_{n-1} - \frac{1}{b_n}\right)}}{q_{n-2} - \dfrac{q_{n-3}}{\left(b_{n-1} - \frac{1}{b_n}\right)}}$$

Factor out $\left(b_{n-1} - \frac{1}{b_n}\right)$

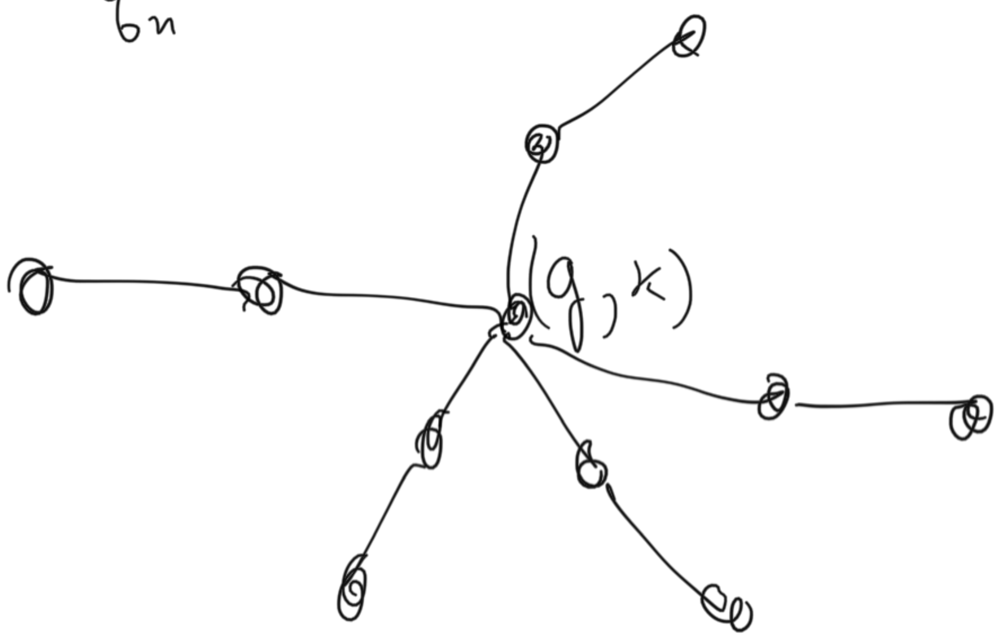
and continue . . .

$$= \frac{p_1 - \cfrac{p_0}{b_2 - \cfrac{1}{b_3 - \cfrac{1}{\ddots \, b_{n-1} - \cfrac{1}{b_n}}}}}{q_1 - \cfrac{q_0}{b_2 - \cfrac{1}{b_3 - \cfrac{1}{\ddots \, b_{n-1} - \cfrac{1}{b_n}}}}}$$

Plug in $p_0 = 1$ $\quad q_0 = 0$

$p_1 = b_1 \quad q_1 = 1$

$$\frac{p_n}{q_n} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots \; b_{n-1} - \cfrac{1}{b_n}}}$$



$\square$