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Normal crossing divisors in symplectic 4-manifolds and affine varieties

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Dedication

To my family.

CHAPTER 1

Introduction and Motivation

There is a close relationship between symplectic geometry and complex geometry. This is especially evident in dimension 4 where a number of different complex and ”algebro-geometric” ideas have been translated to and studied in the symplectic context. Some notable examples are the symplectic blow-up construction, as well as McDuff's characterization of rational and ruled symplectic 4-manifolds [**McD90**], and Donaldson's introduction of the notions of Lefschetz fibrations/pencils [**Don99**] and divisors [**Don96**] in the symplectic category. Though there are many similarities between the two geometries, where these similarities end and where the differences begin is still a subject of great mystery.

In [**Thu76**], Thurston presented the first example of a closed symplectic manifold that is not Kähler. Before the explication of this example, it was not known whether there were any significant differences at all. Thurston's example is 4-dimensional and was first studied in a separate context by Kodaira [**Kod64**] who showed that it admits an integrable almost complex structure and is thus a complex manifold. Because of this, the example has been known in the literature as the *Kodaira-Thurston manifold* which we will denote by M_{KT}. In Kodaira's original presentation, we have $M_{KT} := S^1 \times Nil^3/H$ where Nil^3 is the Heisenberg group and $H \subset Nil^3$ is a lattice. We will adopt Thurston's presentation as the product $M_{KT} := S^1 \times \mathcal{M}^3_\phi$ where $\mathcal{M}^3_\phi$ is the *mapping torus* of a right handed Dehn twist $\phi\colon T^2 \to T^2$ which is the 3–manifold defined by the quotient

$$\mathcal{M}^3_\phi := T^2 \times I / \sim$$

where $(0, p) \sim (1, \phi(p))$. The diffeomorphism type of $\mathcal{M}_\phi$ is independent of the curve about which we perform the Dehn twist via an isotopy.

The manifold M_{KT} cannot be Kähler because we have $b_1(M_{KT}) = 3$ [?Thurston:example]. Kähler manifolds possess a natural Hodge structure and their first Betti number is always even [**GH78**, Chapter 7].

In [**Gom95**], Gompf showed that, for even dimensions $2n \geq 4$, any finitely presented group G can be made the fundamental group of some closed non-Kähler symplectic $2n$-manifold. Gompf's construction presents an avenue for finding many non-Kähler examples using obstructions similar to Thurston's. With all of this in mind, it is clear that being Kähler imposes restrictions on the underlying symplectic geometry, at least on the level of fundamental groups and first homology.

Similar to the problem of determining which symplectic manifolds are Kähler is the problem of determining whether a given non-compact symplectic manifold can be symplectomorphic to a complex *affine variety*. Here we take the classical perspective and consider an affine variety to be the mutual vanishing locus of some ideal in a complex polynomial ring. In [**McL18**], McLean constructs an infinite family of symplectic $\mathbb{C}^n$s, none of which are symplectomorphic to the standard affine $\mathbb{C}^n$. The construction works in any complex dimension $n \geq 4$. All of McLean's examples are of course contractible which tells us that naive cohomological techniques are unlikely to produce an adequate obstruction to being affine. In [**McL12**], via an approach first introduced by Seidel [**Sei08**], McLean established a more sophisticated obstruction relying on the *growth rate of symplectic homology* (c.f. [**Sei08**]). Through use of this obstruction, McLean has been able to determine, for example, when a cotangent bundle may be symplectomorphic to an affine variety [**McL12**, Corollary 1.3].

The growth rate techniques of McLean and Seidel take a view toward a highly relevant phenomenon common to all affine varieties: if $X^{2n} \subset \mathbb{C}^N$ is a complex affine variety, we may always *projectivize* X to obtain a (possibly singular) projective variety $\widetilde{M} \subset \mathbb{CP}^N$, its projective completion. By construction, the intersection of the projective completion with an affine chart in $\mathbb{CP}^N$ is biholomorphic to our original affine variety X. The set $\widetilde{D} = \widetilde{M} - X$ is a subvariety of $\widetilde{M}$ called a *compactifying divisor*. It may equivalently be written as the intersection $\widetilde{D} = \widetilde{M} \cap \mathbb{CP}^{N-1}_\infty$ of $\widetilde{M}$ with the projective subspace $\mathbb{CP}^{N-1}_\infty \subset \mathbb{CP}^N$ lying "at infinity."

It is a classical result of Hironaka [**Hir64a**, **Hir64b**] that the singularities of $\widetilde{M}$ can be completely resolved so as to produce a smooth projective variety M. This resolution process is carried out by blowing up points lying on the compactifying divisor and can be performed in such a way that the resulting manifold M still contains a biholomorphic copy of X. The complement $D := M - X$ is a subvariety with particularly nice singularities: (simple) normal-crossing singularities. The main topological conclusion we can draw from this story is summarized by:

Fact: Every affine variety X admits a smooth (projective) compactification M by a normal crossing divisor.

McLean and Seidel's techniques essentially determine whether this sort of compactification is even possible for a given open symplectic manifold X. One would naively hope that the existence of such a compactification is sufficient for X to be symplectomorphic to some affine variety. Our results will demonstrate that this is not the case. Using the results contained in this work, we will construct a non-affine symplectic 4-manifold X_{KT} as the complement of a symplectic normal crossing divisor in the Kodaira-Thurston example.

Theorem 1.0.1. *Let M_{KT} be the Kodaira-Thurston manifold (see* **[Thu76]***). There exists a concave SNC^+ divisor (Section 3.5) denoted by D_{KT} obtained from the image of a generic section union a fiber of M_{KT} thought of as a T^2-bundle. Then $X_{KT} = M_{KT} - D_{KT}$ is not symplectomorphic to any affine variety.*

This example was chosen to have a very particular presentation. We have written X_{KT} as the complement (in M_{KT}) of a symplectic manifold D_{KT} with nodal singularities (henceforth known as an SNC^+ divisor). This is the symplectic version of the situation above as it is analogous to possessing a pair (M, D) where M is a smooth projective variety, $D \subset M$ is a normal crossing divisor (in the sense of complex geometry) and $X = M - D$ is a smooth affine variety. We refer (M, D) as a *(projective) algebraic compactification* of the affine variety X by a normal crossing divisor.

Let (M, D) be an algebraic compactification for a smooth affine variety X of real dimension 4 (complex dimension 2). All smooth projective varieties are Kähler and so the Kähler form ω of M gives it the structure of a compact symplectic manifold. We may restrict this symplectic structure to the smooth components of D and thus we may present D as a symplectic surface configuration (an SNC^+ divisor) with X as its compliment in M. When considered from a purely symplectic perspective (forgetting the algebraic structure entirely) we refer to (M, D) as an SNC^+ divisor compactification of the non-compact symplectic manifold X. Unless otherwise specified, we will assume that any pair (M, D) refers to a general SNC^+ divisor compactification (not necessarily an algebraic one). From this point of view, it follows that (M_{KT}, D_{KT}) is an SNC^+ divisor compactification of X_{KT} by definition. Since every affine variety possesses such a compactification, our theorem shows that this presentation, while necessary, is not sufficient to conclude that X is affine. We will write

$$M = \overline{X} \cup_{\partial \overline{X}} N_D$$

where $\overline{X}$ is a compact symplectic manifold with boundary with the same homotopy type as X and N_D is a regular neighborhood of D. More specifically, X is symplectomorphic to a *completion* of $\overline{X}$ as a Liouville domain, (c.f. **[Eli]**). Any other SNC^+ compactification $(\widetilde{M}, \widetilde{D})$ of X has a similar decomposition

$$\widetilde{M} = \widetilde{X} \cup_{\partial \widetilde{X}} N_{\widetilde{D}}$$

and we always have that $\overline{X}$ and $\widetilde{X}$ are symplectic deformation equivalent. This in particular implies that we have $\partial \overline{X}$ diffeomorphic to $\partial \widetilde{X}$ and so we may assume that the decomposition of $\widetilde{M}$ is of the form

$$\widetilde{M} = \overline{X} \cup_{\partial \overline{X}} N_{\widetilde{D}}.$$

From this perspective, understanding the totality of all such compactifications is reduced to understanding which divisors D admit a neighborhood N_D with boundary diffeomorphic to $\partial \overline{X}$ and how the neighborhood N_D is glued to $\overline{X}$.

Each SNC^+ divisor has an associated decorated graph Γ_D which encodes topological information about the divisor. We call Γ_D the *divisor graph* of D. In the case when X is affine and (M, D) is an algebraic compactification, we are provided with some additional geometry: the boundary $\partial \overline{X}$ supports a natural *contact structure* ξ_D. This is not always true for general SNC^+ compactifications. Since we are primarily interested in explicating the difference between algebraic compactifications and general SNC^+ compactifications, we will assume that $\partial \overline{X}$ always supports such a contact structure ξ_D. For our purposes, we will only consider the class of *concave* divisors (see Section 3.5). This class of compactifications can be characterized (see Section 3.5) and $(\partial \overline{X}, \xi_D)$ is always contactomorphic to $(\partial \overline{X}, \xi_{\widetilde{D}})$ for any two compactifications (M, D) and $(\widetilde{M}, \widetilde{D})$. The main question we are interested in is:

Question: How much about the isomorphism type of Γ_D can we learn from the contactomorphism type of $\partial\overline{X}$?

We can see easily that answering this question makes steps toward understanding all SNC^+ compactifications of X. Any such compactification (M, D) has an implicit contactomorphism $\Psi\colon (\partial\overline{X}, \xi_D) \to (\partial N_D, \xi_D)$ which defines the gluing between $\overline{X}$ and N_D. If we were able to discern information about Γ_D purely from this gluing map, we may be able to restrict the class of SNC^+ divisors D that may compactify X. In other words, we would understand something about the structure of the collection of all compactifications of X.

If one neglects symplectic, contact, and algebraic structures entirely and focuses only on the topological aspects of the above discussion, this perspective is well understood. The 3-manifolds $Y_D := \partial N_D$, henceforth known as *divisor boundaries*, are examples of *3-manifold plumbings* which are classical objects in low dimensional topology. The graphs Γ_D are examples of *plumbing graphs* which give a combinatorial description of how to perform a 3-manifold plumbing and construct Y_D topologically. In our discussion, we shall use "divisor graphs" and "plumbing graphs" essentially interchangeably with a preference toward "divisor graph." We will distinguish the two wherever necessary.

In [**Neu81**], it was shown that divisor graphs can be "reduced" to simpler divisor graphs via a well-defind procedure without changing the diffeomorphism type of its associated divisor boundary. This reduction procedure always terminates and we are left with a graph Γ_D^{Top} which we will call the *topological reduction* of Γ_D. Neumann proved that, for most 3-manifold plumbings, the diffeomorphism type of the plumbing characterizes the isomorphism type of Γ_D^{Top}. Rephrased in terms of divisor boundaries, this theorem reads:

Theorem 1.0.2 ([**Neu81**], Theorem 4.2)**.** *Let Y_D and $Y_{\widetilde{D}}$ divisor boundaries with divisor graphs Γ_D and $\Gamma_{\widetilde{D}}$. Then Y_D and $Y_{\widetilde{D}}$ are diffeomorphic if and only if $\Gamma_D^{Top} \approx \Gamma_{\widetilde{D}}^{Top}$.*

Throughout, we shall refer to this result as "Neumann's theorem." While this result does answer the question topologically, it is less useful for our purposes since Neumann's reduction procedure has no regard (and actually may destroy) the contact topology at hand. To deal with this, we develop a new reduction procedure which terminates and produces a (possibly distinct) graph Γ_D^{ξ}, the *contact reduction* of Γ_D. In the case when the divisor boundary associated to Γ_D is a *prime* 3-manifold (c.f. [**Hat**], Chapter 1), by comparing Γ_D^{ξ} and Γ_D^{Top}, we can arrive at some topological conclusions:

Theorem 1.0.3. *Let (M, D) be a concave SNC^+ divisor compactification of X. Let $(\partial_\infty X, \xi)$ denote the ideal contact boundary of X (Section 2.3). Then if Y_D is prime and $\Gamma_D^{\xi} \not\approx \Gamma_D^{Top}$, we have one or more of the following:*

(i) M is a blow-up of a ruled symplectic manifold,

(ii) the contact 3-manifold $(\partial_\infty X, \xi)$ is Seifert-fibered over $\mathbb{RP}^2$ via a fibration with at most one singular fiber, or

(iii) the contact 3-manifold $(\partial_\infty X, \xi)$ *contains an embedded incompressible Klein bottle.*

These conclusions are essentially implied from our main structural theorem for divisor neighborhoods.

Theorem 1.0.4. *Let* N_D *be a concave divisor neighborhood (Chapter 3). Then if* ∂N_D *is a prime 3-manifold and* $\Gamma_D^\xi \not\approx \Gamma_D^{Top}$*, one or more of the following are true:*

(i) We may blow up N_D *to a concave* SNC^+ *divisor neighborhood* $N_{\widetilde{D}}$ *which contains a smooth component* S *diffeomorphic to a sphere with* $S \cdot S = 0$*,*

(ii) $(\partial N_D, \xi_D)$ *is Seifert-fibered over* $\mathbb{RP}^2$ *via a fibration with at most one singular fiber, or*

(iii) $(\partial N_D, \xi_D)$ *contains an embedded incompressible Klein bottle.*

This structural theorem, in turn, is implied by our main results for divisor boundaries (which are implicit in this result if one restricts to statements only about ∂N_D). The details of this will be discussed in Chapter 6.

Definition 1.0.5. We say a divisor boundary (Y_D, ξ_D) is *obstructed* if $\Gamma^\xi \not\approx \Gamma^{Top}$ and *unobstructed* otherwise.

In the unobstructed case, the following result follows directly from [**Neu81**].

Theorem 1.0.6. *Let* (Y_D, ξ_D) *and* $(Y_{\widetilde{D}}, \xi_{\widetilde{D}})$ *be a pair of unobstructed divisor boundaries with divisor graphs* Γ_D *and* $\Gamma_{\widetilde{D}}$*. Suppose that* (Y_D, ξ_D) *and* $(Y_{\widetilde{D}}, \xi_{\widetilde{D}})$ *are contactomorphic. Then* $\Gamma_D^\xi \approx \Gamma_{\widetilde{D}}^\xi$*.*

Thus Theorem 1.0.4 and Theorem 6.2.3 give a complete topological characterization of the extent to which Neumann's theorem fails to hold in the contact category. The proof of Theorem 1.0.1 essentially follows from a corollary of Theorem 6.2.3. In terms of divisor neighborhoods, we have:

Theorem 1.0.7. *Let (N_D, ω) and $(N_{\widetilde{D}}, \widetilde{\omega})$ denote concave divisor neighborhoods (Chapter 3) and suppose that $(\partial N_D, \xi_D)$ and $(\partial N_{\widetilde{D}}, \xi_{\widetilde{D}})$ are their associated divisor boundaries. Suppose that both D and D are unobstructed. Then if $(\partial N_D, \xi_D)$ is contactomorphic to $(\partial N_{\widetilde{D}}, \xi_{\widetilde{D}})$, the symplectic manifolds (N_D, ω) and $(N_{\widetilde{D}}, \widetilde{\omega})$ are diffeomorphic up to blow-ups and blow-downs.*

The majority of the work in verifying that X_{KT} is not affine is reduced to verifying that the divisor boundary associated to (M_{KT}, D_{KT}) is unobstructed in the above sense. From there we can use Theorem 1.0.4 to understand all concave SNC^+ divisor compactifications of X_{KT} which will allow us to conclude Theorem 1.0.1. The details of this will be handled in the latter part of Chapter 6.

There is also a subtlety involving mapping classes of gluings that is also dealt with. The most general version of our results in the context of symplectic 4-manifolds may be summarized as:

Theorem 1.0.8. *Let D be a concave compactifying SNC^+ divisor for a 4-dimensional Liouville domain (X^4, ω) and let (M^4, ω) be a compactification of X by D with associated mapping class $[\Psi] \in \pi_0(Cont(\partial X))$ defining the capping. If D is obstructed and ∂X is a prime 3-manifold, then at least one of the following are true:*

(i) (M, ω) *is a blow-up of a ruled symplectic manifold,*

(ii) ∂X *is Seifert-fibered over* $\mathbb{RP}^2$ *via a fibration with at most one singular fiber,*

(iii) ∂X *contains a virtually essential Klein bottle.*

Otherwise, if D is unobstructed, any other SNC^+ *divisor compactification* $(\widetilde{M}, \widetilde{\omega})$ *with the same mapping class* $[\Psi]$ *is diffeomorphic to* (M, ω) *up to blow-ups and blow-downs.*

The outline of the paper is as follows: We will start with some necessary preliminaries relevant to our three geometries: symplectic, contact, and complex algebraic (Chapter 2). We will then outline the topological and some of the symplectic/contact geometric theory of symplectic normal crossing (or SNC^+) divisors (Chapter 3). We then turn to the classical theory of plumbings and graph manifolds which make up the main theoretical context for proving our result (Chapter 4). We will outline a framework and process for normalizing divisor graphs while preserving the contact structure on the divisor boundary (Chapter 5). After showing that we can almost normalize any divisor graph, we will collect all the topological ramifications into the main results of the paper (Chapter 6). We will then focus our attention to the particular case (M_{KT}, D_{KT}) above. We will use Theorem 1.0.3 to draw conclusions about X_{KT} and D_{KT} that will allow us to conclude with Theorem 1.0.1. A short discussion on birational symplectic geometry will follow.

CHAPTER 2

Preliminaries

This section collects short definitions and descriptions of the geometric objects relevant to the sections that follow. Everything will be phrased in low dimensions as we will only focus on 4-dimensional varieties and symplectic manifolds with 3-dimensional contact-type boundaries. As a result, the divisors we concern ourselves with are all symplectic surface configurations.

2.1. Symplectic Geometry

We are primarily motivated by the geometry and topology of symplectic 4-manifolds. A 4-manifold M^4 is *symplectic* if it admits a 2-form $\omega \in \Omega^2_{dR}(M)$ such that

- $d\omega = 0$
- for every $p \in M$ and $0 \neq v \in T_pM$, there exists $w \in T_pM$ such that $\omega(v, w) \neq 0$.

These properties respectively mean that ω is closed and non-degenerate. Symplectic manifolds often have topological properties as if they were simply smooth manifolds but also can express a great deal of rigidity. Because of the first condition above, the gluing of two symplectic manifolds along a common boundary may not be symplectic since one would have to additionally show that the symplectic forms can be completed to a closed form on the gluing.

2.2. Contact Geometry

A contact structure on a 3-manifold Y^3 is a 2-plane distribution $\xi \subset TY$ which is locally given by the kernel of a 1-form $\alpha \in \Omega^1_{dR}(Y)$ such that

$$\alpha \wedge d\alpha \neq 0 \qquad (2.2.1)$$

The condition (2.2.1) is equivalent to saying that $d\alpha|_\xi$ is a symplectic form. There is a unique vector field R satisfying

- $\alpha(R) = 1$
- $i_R d\alpha = 0.$

This vector field is called the *Reeb vector field* associated to α and it spans a 1-dimensional distribution isomorphic to TY/ξ. This distribution and ξ are oriented, respectively, by R and the symplectic form $d\alpha|_\xi$. These orientations combine to give a local orientation wherever α is defined. It follows straight from the definition that $\alpha \wedge d\alpha \neq 0$ is a compatible local volume form. A contact manifold (Y, ξ) is co-orientable if we can extend these local orientations to a global orientation of Y. If Y is already oriented, we then say that ξ is *positive* if the local volume forms $\alpha \wedge d\alpha$ are positively oriented and we say it is *negative* otherwise.

2.3. Liouville domains and manifolds

A *Liouville domain* is an exact symplectic manifold with boundary $(N, d\lambda)$ with $\lambda \in \Omega^1_{dR}(N)$. Via the isomorphism between 1-forms and vector fields on N induced by the symplectic form $d\lambda$, there exists a unique vector field V_λ such that $d\lambda(V_\lambda, W) = \lambda(W)$ for any vector field W. The vector field V_λ is called the *Liouville vector field* associated to the primitive λ. In most definitions found in the literature, V_λ is required to be transverse to ∂N. The domain $(N, d\lambda)$ is called *concave* if V_λ points inward along ∂N and *convex* if it points outward.

The Liouville vector field induces a contact structure ξ on the boundary $Y := \partial N$. Since V_λ is transverse to the boundary and $\lambda(V_\lambda) = 0$, $\alpha := \lambda|_Y$ must be a non-vanishing 1-form. This follows from the fact that $\lambda(W) = d\lambda(V_\lambda, W)$ and the fact that $d\lambda$ is a symplectic form. Additionally,

$$i_{V_\lambda}(d\lambda \wedge d\lambda) = 2\lambda \wedge d\lambda \neq 0$$

so $\alpha \wedge d\alpha$ is a volume form on Y. This implies that $\xi := ker(\alpha)$ is a contact structure. The contactomorphism type of (Y, ξ) is independent of choice of primitive λ with outward/inward pointing Liouville field.

We can complete a Liouville domain $(N, d\lambda)$ to a non-compact exact symplectic manifold without boundary $(X, d\lambda)$ by attaching a cylindrical end modeled off of the symplectization of $(Y, \xi = ker(\alpha))$. Recall that the *(positive/negative) symplectization* of the contact manifold (Y, ξ) with respect to the contact form α is the manifold

$$Symp_{\pm}(Y,\xi) := Y \times \mathbb{R}$$

endowed with the symplectic form $\omega := d(e^{\pm t}\alpha)$. Let $End_{\pm}(N,\omega) := Y \times \mathbb{R}_{\geq 0}$ and define

$$X := N \cup_{\Psi} End_{\pm}(N, d\lambda)$$

by gluing either $End_{-}(N, d\lambda)$ or $End_{+}(N, d\lambda)$ to N along a symplectomorphism of Ψ of collar neighborhoods that sends V_{λ} to the vector field $\pm\frac{\partial}{\partial t}$. If N is convex, we choose $End_{+}(N, d\lambda)$ and if N is concave, we choose $End_{-}(N, d\lambda)$. The result is called the *completion* of the Liouville domain $(N, d\lambda)$.

A *Liouville manifold* is an exact symplectic manifold $(X, d\lambda)$ which is symplectomorphic to the completion of a Liouville domain. Given such a manifold, we have an associated contact manifold (Y, ξ) as above. This contact manifold is called *the ideal contact boundary* or *the boundary at infinity* of $(X, d\lambda)$ and is typically denoted $(\partial_{\infty} X, \xi)$ to emphasize its connection to $(X, d\lambda)$.

2.4. Divisors in algebraic geometry

We assume the reader has some familiarity with classical algebraic geometry. We are primarily interested in the symplectic geometry of classical affine varieties. As an example, we consider an affine hypersurface: let $\mathfrak{I} \subset \mathbb{C}[x, y, z]$ be a polynomial ideal and let $X := V(\mathfrak{I}) \subset \mathbb{C}^3$. Here we define:

$$V(\mathfrak{I}) := \{p \in \mathbb{C}^3 : h(p) = 0,\ h \in \mathfrak{I}\}$$

The variety X being an affine hypersurface simply means that $\mathfrak{I}$ is generated by a single polynomial

$$\mathfrak{I} := \langle f \rangle.$$

We are often interested in studying the *divisors* of an algebraic variety (affine or otherwise). For our purposes, we will take the following definition:

Definition 2.4.1. Let M be an algebraic variety. A *divisor* is a (complex) co-dimension one sub-variety of M.

Remark 2.4.2. For algebraic surfaces (ambient complex dimension 2), co-dimension one sub-varieties correspond to algebraic curves i.e. real 2-dimensional surfaces.

Every affine variety X admits a *projective completion*, a projective variety $\widetilde{M}$ containing a biholomorphic copy of X. For an affine hypersurface, the completion is obtained as the vanishing locus of the *homegenization* of the defining polynomial. For our example above, this is a homogeneous polynomial $F\colon \mathbb{C}^4 \to \mathbb{C}$ given by

$$F(x, y, z, w) = w^{deg(f)} f\left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}\right).$$

This polynomial defines a well defined projective variety as a subset of $\mathbb{CP}^3$, complex projective 3-space, given in homogeneous coordinates by

$$V(F) = \{[x : y : z : w] \in \mathbb{CP}^3 : F(x, y, z, w) = 0\}.$$

The subspace $\widetilde{M} \cap \{w = 1\} \subset \mathbb{CP}^3$ is biholomorphic to X. The space $D := \widetilde{M} \cap \{w = 0\}$ is an algebraic subvariety of $\widetilde{M}$ and we have that $M - D$ is also biholomorphic to X. We call D the *divisor at infinity* or the *compactifying divisor* for X. As mentioned above, via Hironaka's results [**Hir64a**, **Hir64b**], we may always assume that we are working with a smooth projective completion whose compactifying divisor has at worst *nodal singularities* which we will now define.

The *standard nodal singularity* is the algebraic variety $X_{Node} := V(xy)$. The variety X_{Node} decomposes into two components

$$X_L = \{y = 0\}$$

$$X_R := \{x = 0\}.$$

An algebraic curve C is *nodal* if its singular set consists of points locally modeled after X_{Node}. If $C \subset X$ is a divisor, then C is said to be a *normal crossing divisor*. Here "normal crossing" refers to the fact that the two components of X_{Node} intersect orthogonally at the singular point.

CHAPTER 3

Symplectic divisors with normal crossings

Our goal is to reason with the symplectic and contact geometry associated with certain singular symplectic submanifolds, symplectic normal crossing divisors. In this section, we will outline the topological and geometric setting relevant to our results.

3.1. ω-orthogonal divisors and regular neighborhoods

Let (M^4, ω) be a symplectic 4-manifold. An embedded closed symplectic surface $\Sigma^2 \subset M^4$ is called a *symplectic divisor*. We want to understand configurations of symplectic divisors that have particularly nice intersection properties:

Two symplectic divisors $\Sigma_L, \Sigma_R \subset M$ are said to intersect ω-orthogonally if, at every point $p \in \Sigma_L \cap \Sigma_R$, we have that

$$\omega_p(v, w) = 0$$

for all $v \in T_p\Sigma_L, w \in T_p\Sigma_R$. This condition ensures that their tangent spaces span linearly independent symplectic subspaces of T_pM. The intersection is "positive" in the sense that bases that are positively oriented with respect to $T_p\Sigma_L$ and $T_p\Sigma_R$ combine to form a positively oriented basis for T_pM.

Our primary concern is with the local geometry of M near such divisors. There are three different aspects to understanding the local theory:

- topology
- symplectic geometry
- contact geometry

We will handle each of these contexts separately and then describe the full perspective.

3.2. The Topological Context

We will start with the topology. We consider the situation of two divisors intersecting transversely and ω-orthogonally at a single point $p \in \Sigma_L \cap \Sigma_R$. To get started, we must have a clear topological description of "regular" tubular neighborhoods of $D := \Sigma_L \cup \Sigma_R$.

We start at the intersection point $p \in \Sigma_L \cap \Sigma_R$. Fix a neighborhood $\mathscr{O}p(p)$ of p diffeomorphic to $\mathbb{C}^2$ whose intersection with Σ_L is diffeomorphic to the coordinate plane $\mathbb{C} \times \{0\}$ and whose intersection with Σ_R is the other coordinate plane $\{0\} \times \mathbb{C}$. This may always be done since Σ_L and Σ_R intersect transversely. We let $U_L, U_R \subset \mathscr{O}p(p)$ denote regular neighborhoods (thickenings of constant radius) of the coordinate planes. The situation is given schematically by the following figure:

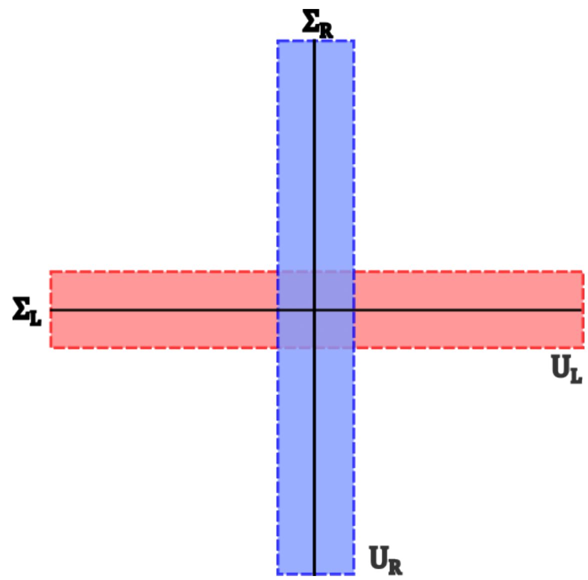


Figure 3.1. Portions of Σ_L and Σ_R in a neighborhood of the intersection

Via the tubular neighborhood theorem, we may choose U_L and U_R such that they may be identified with neighborhoods of portions of the zero sections of the normal bundles $\nu(\Sigma_L)$ and $\nu(\Sigma_R)$. By extending these portions to full tubular neighborhoods of the zero sections, we may extend U_L and U_R to full tubular neighborhoods

$$E_L \subset \nu(\Sigma_L),$$

$$E_R \subset \nu(\Sigma_R).$$

We choose these neighborhoods such that $E_L \cap E_R = U_L \cap U_R$. We refer to the union

$$N_D := E_L \cup E_R$$

as a *regular neighborhood of* D.

3.2.1. Regular neighborhoods as plumbings

By definition, E_L and E_R are diffeomorphic to the total spaces of a pair of $\mathbb{D}^2$-subbundles of the normal bundles $\nu(\Sigma_L), \nu(\Sigma_R)$. Thought of this way, the individual bundles have associated *Euler numbers*

$$k_L := \frac{1}{2\pi} \int_{\Sigma_L} e(E_L)$$

$$k_R := \frac{1}{2\pi} \int_{\Sigma_R} e(E_R)$$

where $e(E_L) \in H^2_{dR}(\Sigma_L)$ and $e(E_R) \in H^2_{dR}(\Sigma_L)$ are the Euler classes of the $\mathbb{D}^2$-bundles E_L, E_R (c.f. **[BT82]**, Chapter 6). Since E_L, E_R are normal disk-bundles of full rank, we know that these numbers are equal to the self-intersection numbers of our two divisors:

$$k_L = [\Sigma_L] \cdot [\Sigma_L]$$

$$k_R = [\Sigma_R] \cdot [\Sigma_R]$$

(c.f. **[Gom95]**, Chapter 4, Example 4.6.5).

The restrictions $E_L|_{U_L \cap U_R}$ and $E_R|_{U_L \cap U_R}$ are $\mathbb{D}^2$-subbundles which lie over disks $\Delta_L \subset \Sigma_L$ and $\Delta_R \subset \Sigma_R$ and so we may trivialize these restrictions. Our $\mathbb{C}^2$-neighborhood provides diffeomorphisms

$$E_L|_{U_L \cap U_R} \approx \Delta_L \times \mathbb{D}^2 \approx \mathbb{D}^2 \times \mathbb{D}^2,$$

$$E_R|_{U_L \cap U_R} \approx \Delta_R \times \mathbb{D}^2 \approx \mathbb{D}^2 \times \mathbb{D}^2.$$

These restrictions are easily seen to be identified in the union along a map $\mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{D}^2 \times \mathbb{D}^2$ which switches the two factors (c.f. [**GS99**], Chapter 4, Example 4.6.2).

This description presents a regular neighborhood topologically as a gluing of two disk bundles over our initial surfaces. The abstract construction associated to this situation is called plumbing: Given two disk bundles $E_L \to \Sigma_L$ and $E_R \to \Sigma_R$, we start by fixing trivializations over portions $\Delta_L \subset \Sigma_L$ and $\Delta_R \subset \Sigma_R$ diffeomorphic to disks. These trivializations give us a pair of poly-disks

$$E_L|_{\Delta_L} \approx \mathbb{D}^2 \times \mathbb{D}^2$$

$$E_R|_{\Delta_R} \approx \mathbb{D}^2 \times \mathbb{D}^2$$

and the bundles are "plumbed" together by gluing these poly-disks together along the map $\mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{D}^2 \times \mathbb{D}^2$ which switches the two factors. The diffeomorphism type of a plumbing is independent of our choice of local trivializations. From this description, it is clear that regular neighborhoods are disk bundle plumbings.

3.2.2. Plumbing boundaries and plumbing surgery

The boundary of a regular neighborhood may be seen as the result of a surgery operation on a pair of S^1-bundles or, equivalently, a gluing operation between S^1-bundles over surfaces with S^1-boundary components.

Continuing with the abstract presentation above, the data we have fixed is a pair of $\mathbb{D}^2$-subbundles

$$\begin{array}{ccc} \mathbb{D}^2 \hookrightarrow & E_L|_{U_R \cap U_R} & \\ & \downarrow & \\ & \Delta_L & \end{array} \quad \begin{array}{ccc} \mathbb{D}^2 \hookrightarrow & E_L|_{U_R \cap U_R} & \\ & \downarrow & \\ & \Delta_R & \end{array}$$

over portions of Σ_L, Σ_R diffeomorphic to disks. We let

$$\Sigma_L^* := \overline{\Sigma_L - \Delta_L}$$

$$\Sigma_R^* := \overline{\Sigma_R - \Delta_R}$$

these are compact surfaces with S^1-boundary. Each of the restricted bundles $E_L|_{\Sigma_L^*}$,$E_R|_{\Sigma_R^*}$,$E_L|_{\Delta_L}$ and $E_R|_{\Delta_R}$ is a smooth manifold with corners. We easily see, however, that the sets

$$N_L := E_L|_{\partial \Sigma_L^*} = E_L|_{\partial \Delta_L}$$

and

$$N_R := E_R|_{\partial \Sigma_R^*} = E_R|_{\partial \Delta_R}$$

are solid tori which lie in portions of the boundaries of these restrictions. From the schematic for our $\mathbb{C}^2$-neighborhood, we see that the interiors of these solid tori both lie in the interior of the regular neighborhood N_D.

The corner locus of each of these restrictions are tori and may be identified with one of the boundary tori of N_L and N_R. In the full regular neighborhood, these tori are all identified and lie in the boundary of the neighborhood. We denote the identified torus by T and refer to it as *a plumbing torus* lying in the boundary of our regular neighborhood.

Cutting ∂N_D along T splits the boundary into two pieces. By construction, these pieces may be identified by portions of the boundary S^1-bundles $\partial E_L|_{\Sigma_L^*}$ and $\partial E_L|_{\Sigma_R^*}$ lying over the interiors of Σ_L^* and Σ_R^*. By taking the closure of these pieces, we obtain two S^1-bundles $A_L \subset \partial U_L$ and $A_R \subset \partial U_R$ over the surfaces Σ_L^*, Σ_R^*:

$$\begin{array}{ccc} S^1 \hookrightarrow A_L & & S^1 \hookrightarrow A_R \\ \downarrow & & \downarrow \\ \Sigma_L^* & & \Sigma_R^* \end{array}$$

In the full boundary ∂N_D, these pieces intersect along the plumbing torus T. This description presents the boundary of the entire regular neighborhood as a gluing of A_L and A_R along a diffeomorphism of their torus boundaries

$$\partial A_L \to \partial A_R.$$

We can understand this gluing as follows:

(i) Using the coordinates provided by the neighborhood of the intersection, we let $F_L := \partial\mathbb{D}^2 \times \{pt\}$ and $F_R := \{pt\} \times \partial\mathbb{D}^2$. This provides our corner locus $T := \partial\mathbb{D}^2 \times \partial\mathbb{D}^2$ with an oriented basis on homology

$$H_1(T) = \mathbb{Z}\langle [F_L], [F_R] \rangle.$$

(ii) Fixing a pair of sections $\sigma_L \colon \Sigma_L^* \to A_L$ and $\sigma_R \colon \Sigma_R^* \to A_R$ defines for us bases for the homology of our boundary tori

$$H_1(\partial A_L) = \mathbb{Z}\langle [\sigma_L|_{\partial A_L}], [F_L] \rangle$$

and

$$H_1(\partial A_R) = \mathbb{Z}\langle [\sigma_R|_{\partial A_R}], [F_R] \rangle.$$

(iii) With all these bases, we have gluing maps $\psi_L \colon \partial A_L \to T$ and $\psi_R \colon \partial A_R \to T$. In matrix notation, these maps act on homology via

$$(\psi_L)_* = \begin{bmatrix} -k_L & 1 \\ -1 & 0 \end{bmatrix}$$

and

$$(\psi_R)_* = \begin{bmatrix} -1 & 0 \\ -k_R & 1 \end{bmatrix}.$$

Here k_L and k_R are the Euler numbers of the respective bundles ∂E_L and ∂E_R since we must have

$$\partial E_L = A_L \cup_{\psi_L} N_R$$

$$\partial E_R = A_R \cup_{\psi_R} N_L$$

The gluing map is identified with $\psi = \psi_R^{-1} \circ \psi_L \colon \partial A_L \to \partial A_R$. The induced map on homology is then given in matrix notation by

$$\psi_* = \begin{bmatrix} -1 & 0 \\ -k_R & 1 \end{bmatrix} \begin{bmatrix} -k_L & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} k_L & -1 \\ k_L k_R - 1 & k_R \end{bmatrix}$$

so this map sends a meridian of A_L to a curve of slope $k_R - \frac{1}{k_L}$.

The abstract construction associated to this story is called plumbing surgery. Given our two $\mathbb{D}^2$-bundles

$$E_L \to \Sigma_L$$

$$E_R \to \Sigma_L$$

as above we consider the S^1-bundles

$$\partial E_L \to \Sigma_L$$

$$\partial E_R \to \Sigma_R$$

Plumbing surgery is performed by removing the interiors of trivializing solid tori from each and then gluing the remaining pieces together along their boundary. As above, we require that, with respect to some set of coordinates, the gluing map sends a meridian of one torus boundary to a curve of slope $k_R - \frac{1}{k_L}$ in the other.

Equivalently, we could have started with a pair of trivial S^1-bundles with boundary

$$\Sigma_L^* \times S^1 \to \Sigma_L^*$$

$$\Sigma_R^* \times S^1 \to \Sigma_R^*$$

together with a pair of integers (k_L, k_R) and glue them together as above. In this situation, the integers k_L and k_R referred to as *Waldhausen framings* of the respective pieces.

3.3. SNC^+ Divisors

Extending the notion of ω-orthogonality beyond two intersecting divisors is straightforward. Throughout, we will consider configurations of multiple mutually ω-orthogonal symplectic divisors. In continuing the analogy between symplectic geometry and algebraic geometry, we have:

Definition 3.3.1. A subspace $D \subset (M^4, \omega)$ is called a (simple) *symplectic divisor with positive normal crossings* (or simply an SNC^+-divisor) if we may write

$$D = \bigcup_{i=1}^{N} D_i$$

where $\{D_i\}_{i=1}^N$ is a collection of embedded closed symplectic surfaces in M such that any pair (D_i, D_j) with $D_i \cap D_j \neq \emptyset$ intersects ω-orthogonally only in transverse double points (i.e. all intersections must be modeled after intersecting coordinate planes in $\mathbb{C}^2$). The manifolds D_i are called the *smooth components* of the divisor D.

It is also straightforward to generalize the notion of a regular neighborhood of a pair of normal crossing divisors to general SNC^+ divisors. We may then readily see a regular neighborhood as the result of plumbing multiple disk bundles together.

To each SNC^+ divisor D, we may construct a finite graph Γ_D encoding the connectivity information of D as follows:

- The vertex set of Γ_D consists of one vertex v_i for every component D_i.
- Two distinct vertices v_i, v_j are connected by an edge e_{ij}^k for each $p_k \in D_i \cap D_j \neq \emptyset$.

It is important to note that, with this definition, there are no edges in Γ_D from a vertex to itself (i.e. Γ_D is free of loops). Thus SNC^+ divisors, as defined above, are topologically identical to simple normal crossing divisors in the complex-algebro-geometric setting, justifying their name.

Remark 3.3.2. For notational convenience, we will re-label the components of D by the corresponding vertices of Γ_D. Any vertex $v \in \Gamma_D$ corresponds to a component $D_v \subset D$. We let $\mathcal{E}(v)$ denote the set of edges in Γ_D adjacent to v and we let $\mathcal{N}(v)$ be the set of vertices neighboring v.

3.4. Divisor boundaries

The boundary of a regular neighborhood of an SNC^+ divisor is called a *divisor boundary* and is denoted by $Y_D := \partial N_D$ where N_D is the neighborhood of D discussed above.

By following a similar procedure as in Section 3.2.2 above, we may describe the topology of Y_D abstractly. First, we now obtain a collection $\mathcal{T}_D := \{T_e\}_{e\in\Gamma_D}$ of *plumbing tori*. Cutting Y_D along these tori splits it into a collection $\{A_v\}_{v\in\Gamma_D}$ of S^1 over a collection of surfaces with boundary $\{\Sigma_v^*\}_{v\in\Gamma_D}$, all defined analogously to Σ_L^* and Σ_R^* in Section 3.2.2. Fixing sections $\sigma_v \colon \Sigma_v^* \to A_v$ as before now yields, for each v, a collection of integers $\{k_{vw}\}_{w\in\mathcal{N}(v)}$ such that

$$k_v = \sum_{w\in\mathcal{N}(v)} k_{vw}$$

which describe the gluing maps for the piece A_v. This collection is called a *Waldhausen framing* for the vertex. We note that choosing a different section may give rise to a different Waldhausen framing.

The abstract description thus starts with a collection $\{\Sigma_v^*\}_{v\in\Gamma_D}$ of surfaces with boundary giving rise to a collection of trivial S^1-bundles $\{\Sigma_v^* \times S^1\}_{v\in\Gamma_D}$. Each fixed surface Σ_v^* is given an associated Euler number k_v and a collection of integers $\{k_{vw}\}_{w\in\mathcal{N}(v)}$ such that

$$k_v = \sum_{w\in\mathcal{N}(v)} k_{vw}$$

and the bundles are glued together along apropriate diffeomorphisms of their boundary components as in Section 3.2.2.

3.5. Liouville domains, concavity, and the GS-criterion

In many cases, closed neighborhoods of SNC^+ divisors will admit the structure of a Liouville domain.

Definition 3.5.1. An SNC^+ divisor D is said to be *concave* if every regular neighborhood N_D admits a closed sub-neighborhood N_D such that $(N_D, \omega|_{N_D})$ is a concave Liouville domain.

The Liouville vector field induces a contact structure ξ on the boundary $Y_D := \partial N_D$. The contactomorphism type of (Y_D, ξ) is independent of choice of primitive λ with inward pointing Liouville field inducing the contact structure.

There is a simple condition that determines when N admits a concave Liouville structure. Every SNC^+ divisor, concave or otherwise, has an associated *intersection form* $Q_D\colon \Gamma_D \times \Gamma_D \to \mathbb{Z}$ defined by

$$Q_D(v,w) = D_v \cdot D_w$$

From this definition, $Q_D(v,w) = d$ if Γ_D has $d \geq 0$ edges from v to w and $Q_D(v,v)$ is the self-intersection number (in M) of D_v. Placing an arbitrary order on the vertices $v_1, \ldots, v_N$, we regard the intersection form as a linear map $Q_D \colon \mathbb{R}^N \to \mathbb{R}^N$ defined, as a matrix, by the entries

$$(Q_D)_{ij} = Q_D(v_i, v_j).$$

We also have a well defined *area vector* $\underline{a} \in \mathbb{R}^N$ with entries

$$a_{v_i} = \int_{D_{v_i}} \omega.$$

Definition 3.5.2. We say that D satisfies *the positive GS-criterion* if there exists a vector $\underline{b} \in \mathbb{R}^N$ with positive entries solving

$$Q_D \underline{b} = \underline{a}.$$

We then have:

Theorem 3.5.3 ([LM19], Theorem 1.1). *If D satisfies the positive GS-criterion, then D has a concave neighborhood N_D inside any regular neighborhood U via the GS-construction (*[**LM**]*, Section 2.1). The contactomorphism type of the boundary of N_D is independent of all choices made during the GS-construction.*

Remark 3.5.4. "GS" refers to the authors of **[GS09]** wherein a separate condition involving negative solutions is shown to be sufficient to ensure the existence of a *convex* neighborhood.

Keeping track of the full symplectic topological data of concave neighborhoods requires decorations on the graph Γ_D in the form of:

- Functions $(g, k)\colon Vert(\Gamma_D) \to \mathbb{N}_{\geq 0} \times \mathbb{Z}$ where $g(v)$ is the genus and $k(v)$ is the self intersection number of the divisor D_v and $Vert(\Gamma_D)$ is the set of vertices of Γ_D.
- The symplectic area vector $\underline{a}$.

For the remainder of our discussion, all divisors will assumed to be concave and so we will assume that their (decorated) graphs satisfy the positive GS-criterion.

CHAPTER 4

Divisor Boundaries, Graph Manifolds, and Topological Plumbing Calculus

The fact that closed regular neighborhoods of concave divisors admit contact type boundaries means that to every such divisor D, we may associate a contact 3-manifold (Y_D, ξ_D). The contactomorphism type of (Y_D, ξ_D) is stable up to perturbation of the non-vanishing Liouville vector field. Any such 3-manifold Y_D is called a *divisor boundary* and the contact manifold (Y_D, ξ_D) is called a *contact divisor boundary*. As previously mentioned, divisor boundaries are examples of 3-manifold plumbings. These objects are classical to topology and have received extensive attention and use.

The diffeomorphism type of Y_D is completely determined by the divisor D. This follows from classical results in smooth topology. We are interested in understanding to what extent the diffeomorphism type of Y_D determines the contactomorphism type of (Y_D, ξ_D).

4.1. The canonical contact structure of a plumbing

Let Γ be an arbitrary plumbing graph with standard genus and self-intersection number decorations (g,k). As before, we let $g(v), k(v)$, and $d(v)$ respectively denote the genus, self-intersection number, and degree of a vertex v. For now, we assume that Γ has more than one vertex.

For each vertex v, we let Σ_v^* be a genus $g(v)$ surface with $d(v)$ boundary components. We fix a labeling of the boundary components of Σ_v^* by the neighboring nodes of v in Γ. We let C_w denote the boundary component corresponding to $w \in \mathcal{N}(v)$. We let $Y_v := \Sigma_v^* \times S^1$ and let $[F_v] \in H_2(Y_v)$ denote the homology class of $F_v := pt \times S^1$ in Y_v.

Choose a collection of closed curves $\gamma_{v,w} \subset C_w \times S^1 \subset Y_v$ such that $[\gamma_{v,w}] \cdot [C_w \times pt] = 1$ and such that the collection of integers $k_w := [\gamma_{v,w}] \cdot [F_v]$ satisfies

$$k(v) = \sum_{w \in \mathcal{N}(v)} k_w.$$

Finally, we let $S_v \subset Y_v$ denote a section of the trivial fibration Y_v such that $\partial_w S_v = \gamma_{v,w}$.

Topologically, the plumbing boundary Y_Γ may be constructed by gluing these pieces together. For any fixed vertex v, we glue Y_v to Y_w for $w \in \mathcal{N}(v)$ along the map which takes $\gamma_{v,w} \mapsto F_w$ and $F_v \mapsto \gamma_{w,v}$

For any volume form Ω on S_v, we may always find a Liouville 1-form λ whose Liouville vector field with respect to Ω points outward along ∂S_v

4.2. Determining Γ_D from Y_D

The diffeomorphism type of a divisor boundary (or even a more general plumbing) does not determine the isomorphism type of the divisor graph Γ_D. Performing a symplectic blow-up on $(N_D, d\lambda)$ at some point $p \in D$ away from the singular points will produce a new symplectic 4-manifold $(N_{\widetilde{D}}, \widetilde{\omega})$ which is a regular neighborhood of a divisor $\widetilde{D}$ which reduces to D after blowing-down the exceptional divisor. Blowing up is a local construction and does not affect the diffeomorphism type of the boundary $\partial N_{\widetilde{D}}$. We thus have:

Proposition 4.2.1. *If Y_D and $Y_{\widetilde{D}}$ are divisor boundaries, then we may have $Y_D \approx_{Diff} Y_{\widetilde{D}}$ with $\Gamma_D \not\approx \Gamma_{\widetilde{D}}$.*

We conclude that we cannot possibly recover Γ_D from the divisor boundary alone unless we somehow deal with situations like the above.

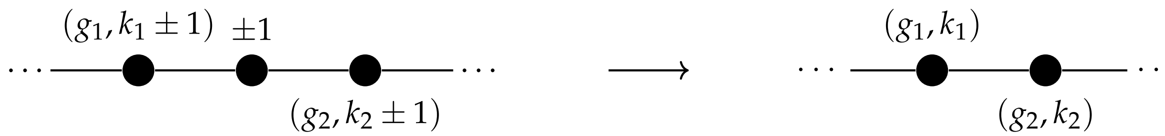
4.3. Topological Plumbing Calculus

Looking at Proposition 4.2.1, we see that the diffeomorphism type of a divisor boundary is not enough to recover the isomorphism type of its associated divisor graph. There is however, still a great deal of rigidity in the relationship between divisor boundaries and their plumbing graphs.

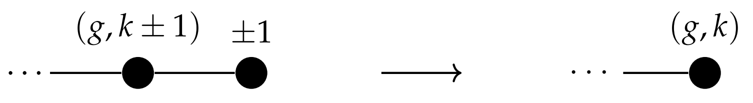
In [**Neu81**], the author defined a collection of eight moves which may be performed on plumbing graphs Γ (see [**Neu81**], Section 2). Each of these moves have associated topological constructions which transform N_Γ into another plumbing $N_{\tilde{\Gamma}}$ while preserving the diffeomorphism type of the boundary.

There are only two moves which are relevant to this manuscript. They are named and defined diagrammatically by:

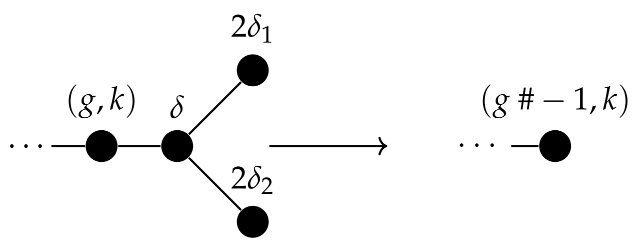
(1) The ± 1 blow-down: Defined by



or



and **(2) $\mathbb{RP}^2$-absorption:** Defined by



Here $\delta_i = \pm 1$, $\delta = \frac{\delta_1+\delta_2}{2}$, and the vertex with decoration $(g \# -1, k)$ denotes the S^1 bundle over $\Sigma_g \# \mathbb{RP}^2$ with Euler number k.

The other moves are (by name only):

(3) 0-curve absorption: This move obviously implies that the graph contains a sphere of self intersection number zero. These graphs will be discussed in the contact setting below.

(4) Unoriented handle absorption: This move is not applicable to SNC^+ divisors as it involves reducing to a graph with a non-orientable vertex.

(5) Oriented handle absorption: This move is equivalent to having a sphere of self-intersection number zero. This situation is dealt with in the contact setting below.

(6) Splitting: All of our divisor graphs will be connected and do not admit any splittings.

(7) Seifert graph exchanges: This performs a number of graph exchanges between components with one vertex and one loop and the standard star-shaped presentations of the Seifert-fibered spaces they represent. Because none of our graphs contain any loops (i.e. edges from a vertex to itself), these exchanges will never be applicable.

(8) Annulus absorption: This move is performed on general plumbings with boundary. Since none of our divisor boundaries have boundaries of their own, this move does not apply.

Definition 4.3.1. Two plumbing graphs Γ_1, Γ_2 are said to be *TPC-combinatorially related*[1] if we can obtain one from the other via a sequence of moves in Neumann's plumbing calculus (see **[Neu81]**, Section 2).

Which brings us to one of the main results of Neumann's paper:

[1]TPC = Topological Plumbing Calculus

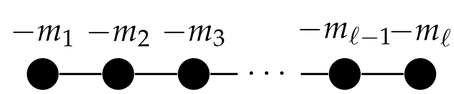
Theorem 4.3.2 ([Neu81], Theorem 3.1). *If Y_Γ is diffeomorphic to $Y_{\widetilde{\Gamma}}$, then Γ and $\widetilde{\Gamma}$ are TPC-combinatorially related.*

From this, we may conclude that the diffeomorphism type of a divisor boundary determines the isomorphism type of its plumbing graph up to topological plumbing calculus. To prove this theorem, Neumann defines what it means for a plumbing graph to be in *normal form* and first shows that every plumbing graph may be put into normal form via topological plumbing calculus. Neumann then shows that the diffeomorphism type of plumbing boundaries whose graphs are in normal form do determine the isomorphism type of its associated graph. In order to write the definition of Neumann's normal form, it helps to discuss a special class of subgraphs relevant to the theorem: chains of spheres.

4.4. Chains of Spheres

A set of plumbing graphs important for our discussion are linear plumbings of spheres or "chains."

Definition 4.4.1. Let Γ be a plumbing graph. A *chain* is a subgraph $C \subset \Gamma$ of the form



with each vertex having degree at most 2 in Γ. The integers $-m_i$ are called *the components of the chain* C. A chain is called *maximal* if it is not contained in a strictly larger chain. A chain is *normal* or *in normal form* if $m_i \geq 2$ for $i > 1$ and m_1 is either 0 or ≥ 2.

Remark 4.4.2. Chains are symmetric and so Y_C is diffeomorphic to $Y_{C'}$ where C' is the chain with components $(-m_\ell, \ldots, -m_1)$.

The following lemma is a classical result in 3-manifold topology. There are many proofs of this result in the literature. The proof given in **[Sym03]** is the very economical and so we reproduce it here.

Lemma 4.4.3. *Let $C \subset \Gamma$ be a chain with components $(-m_1, \ldots, -m_\ell)$ and $m_\ell \neq 0, 1$. Then Y_C is diffeomorphic to a Lens space $L(p, q)$ where*

$$\frac{p}{q} = m_1 - \cfrac{1}{m_2 - \cfrac{1}{\ddots - \frac{1}{m_\ell}}}. \tag{4.4.4}$$

Proof. The topology of the 3-manifold associated to a chain may be written as a union

$$\mathbb{D}^2 \times S^1 \cup_\Phi \mathbb{D}^2 \times S^1$$

where $\Phi \colon T^2 \to T^2$ is a gluing diffeomorphism. We may see this because we may decompose the boundary associated to a chain of spheres into a union

$$\mathbb{D}^2 \times S^1 \cup_{T^2} I \times S^1 \times S^1 \cup_{T^2} \cdots \cup_{T^2} I \times S^1 \times S^1 \cup_{T^2} \mathbb{D}^2 \times S^1$$

by cutting along all plumbing tori. From here, we see that we may use the fact that a manifold glued to a copy of $I \times S^1 \times S^1 \cup_{T^2} \mathbb{D}^2 \times S^1$ is diffeomorphic to a manifold glued to a copy of $\mathbb{D}^2 \times S^1$ along some other diffeomorphism. Applying this to the above composition yields our initial decomposition. The map Φ is simply the composition of all gluing maps involved thought of as diffeomorphisms of T^2.

By using the natural basis on $\partial\mathbb{D} \times S^1$, this may be thought of as an element of $SL(2,\mathbb{Z})$ (c.f. **[Neu81]**, Section 5). The matrix $A_\Phi \in SL(2,\mathbb{Z})$ associated to this gluing is given by the composition

$$A_\Phi := A_1 A_2 \cdots A_L$$

where

$$A_\ell = \begin{bmatrix} -1 & 0 \\ m_\ell & 1 \end{bmatrix}$$

and

$$A_i = \begin{bmatrix} -1 & 0 \\ m_i & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

for $1 \leq i < \ell$. We then have

$$A_1 \cdots A_\ell = \begin{bmatrix} -q & q' \\ p & p' \end{bmatrix}$$

To compute p and q, it suffices to find the coefficients of the image of the meridian $\partial\mathbb{D} \times \{pt\}$. Under A_ℓ, the image is a $(-1, m_\ell)$ curve. For the A_i, we see that the image of a $(-x, y)$ curve under A_i is a $(-y, -x + c_i y)$ curve and so a curve of slope $\frac{y}{x}$ is taken to a curve of slope $m_i - \frac{1}{y/x}$. The result follows by induction. $\square$

From now on, we introduce the following notational conventions:

- We let $[m_1, \ldots, m_\ell]$ denote the finite continued fraction with components $(m_1, \ldots, m_\ell)$ as in the right-hand side of 4.4.4.
- We let $C(m_1, \ldots, m_\ell)$ the chain with components $(-m_1, \ldots, -m_n)$.
- We let $L(m_1, \ldots, m_\ell)$ denote the lens space $L(p, q)$ with $p/q = [m_1, \ldots, m_\ell]$.

In this notation, Lemma 4.4.3 becomes

$$Y_{C(m_1,\ldots,m_\ell)} \approx_{\text{Diff}} L(m_1, \ldots, m_\ell).$$

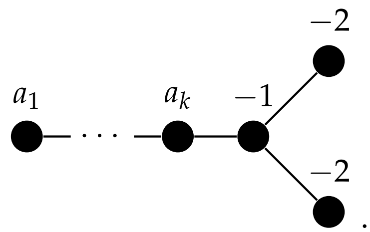
Each positive rational number r has a unique normal chain $C(m_1, \ldots, m_\ell)$ associated to it where $r = [m_1, \ldots, m_\ell]$. This follows directly from the fact that continued fraction expansions of rational numbers are unique if all the components m_i for $i > 1$ satisfy $m_i \geq 2$. Thus every normalized chain is homemorphic to a lens space $L(p, q)$ where p, q are relatively prime and $p/q = r$.

4.5. TPC normal form

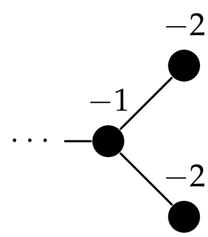
We are now able define Neumann's normal form.

Definition 4.5.1 ([Neu81], Section 4). We say that a plumbing graph Γ is *in TPC normal form* if the following criteria are met:

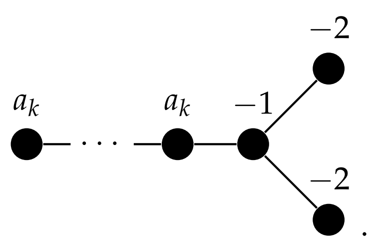
1. No operation from topological plumbing calculus may be performed, except that Γ may have a component of the form:



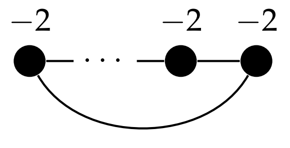
2. The weights e_i on all chains of Γ satisfy $e_i \leq -2$.
3. No portion of the graph has the from



unless it is in a component of Γ of the form



4. No component of Γ is isomorphic to:



In the context of divisor graphs, (4) is a special case of a *circular spherical divisor graph*. Such divisors and their graphs were classified in their entirety in **[LM]** and are not relevant to this paper.

Neumann's major results about graphs in normal form result in a proof of

Theorem 4.5.2 (**[Neu81], Theorem 3.1**). *If Y_D is diffeomorphic to $Y_{\widetilde{D}}$, then Γ_D and $\Gamma_{\widetilde{D}}$ are TPC-combinatorially related.*

It is first proven that:

Lemma 4.5.3 (**[Neu81], Theorem 4.1**). *Any plumbing graph can be reduced to normal form using topological plumbing calculus.*

One then only needs to show:

Theorem 4.5.4 (**[Neu81], Theorem 4.2**). *Let $\Gamma, \widetilde{\Gamma}$ be two plumbing graphs in topological normal form. Then Y_Γ is diffeomorphic to $Y_{\widetilde{\Gamma}}$ if and only if Γ and $\widetilde{\Gamma}$ are isomorphic.*

To prove Lemma 4.5.3, Neumann outlines a procedure for using the topological plumbing calculus to normalize a plumbing graph. Crucial to this procedure is an algorithm for normalizing chains. As we will see in the next section, this algorithm cannot be followed directly in the contact setting while preserving the contactomorphism type of the divisor boundary.

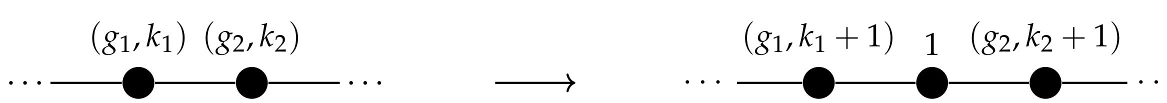
CHAPTER 5

Contact Plumbing Calculus

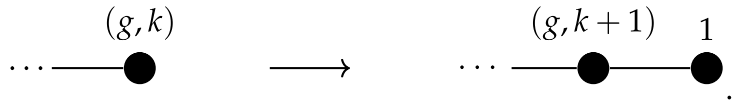
We will see that not every move in the topological plumbing calculus respects the contact topology of the boundary. After determining which moves in the calculus are capable of being carried out in the contact setting, we will augment the calculus with new moves. These moves indeed are already possible within Neumann's calculus but nonetheless have their own unique usage which warrants their own names.

5.1. Contact-Sensitive TPC Moves

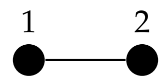
Theorem 4.5.2 tells us that if Y_D and $Y_{\tilde{D}}$ are diffeomorphic, then their divisor graphs are TPC-related. Since contactomorphism implies diffeomorphism, this result clearly also holds with respect to contactomorphism. The issue is that, while all associated topological constructions in Neumann's topological plumbing calculus preserve the diffeomorphism type of the divisor boundary, the moves may change the contactomorphism type of a divisor boundary. An example of a move that is not contact-preserving is the *interior/exterior* $+1$ *blow-up/blow-down* (**[Neu81]**, Move 1) defined by



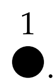
or



The following example demonstrating this was first given in [**LM19**] Example 2.21:

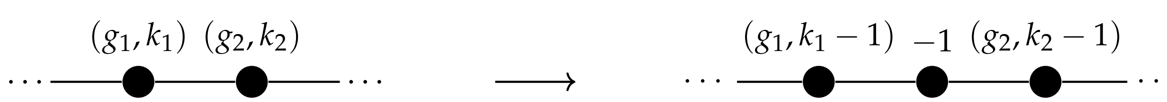


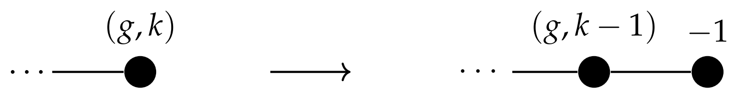
which is the $+1$ blow-up of



The contact 3-manifold associated to the second graph is (S^3, ξ_{std}) but the contact structure associated to the first is overtwisted. Indeed, if the contact structure associated to the first graph were tight, then it would be contactomorphic to the standard S^3 and, by [**Eli92**, Theorem 1.4.1], it would admit a filling by the standard symplectic 4-ball since S^3 is universally tight. It follows that we may fill in a concave neighborhood to a closed symplectic manifold (M, ω). Since M contains a sphere of self-intersection number 1, by [**McD90**, Corollary 1.6], we know that it is a blow-up of $\mathbb{CP}^2$. This tells us that the intersection form of M is diffeomorphic to $\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$ and so $b_2^+(\mathbb{CP}^2) = 1$. From the first graph, we see that we must have $b_2^+(M) = 2$ and so we cannot have such a filling.

The *interior/exterior* -1 *blow-up/blow-down* defined by

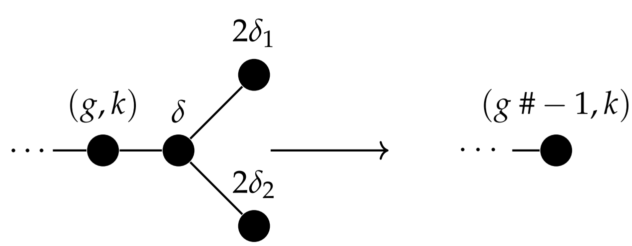




is contact-preserving. This was shown in [**LM**] using methods from toric geometry. Their construction exploits the toric structure of a neighborhood of the intersection of two smooth divisors. A toric diagram for a portion of a concave neighborhood can be obtained near the intersection points and the -1 blow-up can be performed in the toric region by modifying this diagram. One can then glue in the 4-manifold associated to the modified diagram by cutting out the toric region in the original neighborhood. This can be done while preserving concavity to yield a concave neighborhood of the symplectic divisor obtained after performing the blow-up.

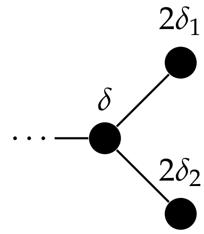
Additionally, many of the moves outlined in topological plumbing calculus are not-applicable as they involve vertices with non-orientable bases which have no direct symplectic analog.

Every component of an SNC^+ divisor is orientable and so any of the moves involving non-orientable surfaces ([**Neu81**], Moves 2 and 4) can be completely removed from consideration with the exception $\mathbb{RP}^2$-*absorption*:



.

Here $\delta_i = \pm 1$, $\delta = \frac{\delta_1+\delta_2}{2}$, and the vertex with decoration $(g\,\# - 1, k)$ denotes the S^1 bundle over $\Sigma_g \# \mathbb{RP}^2$ with Euler number k. This move has no clear symplectic analog but subtlety appears in this story due to the simple fact that the manifold obtained from



by cutting at the plumbing torus corresponding to the leftmost edge is diffeomorphic to an orientable tubular neighborhood of a Klein bottle. This subtlety will be expanded upon in Section 5.5.

Edges from a vertex to itself never appear in SNC^+ plumbing graphs by the *simplicity* condition (i.e. all the components of D must be smooth manifolds). This allows us to discard any moves involving loops ([**Neu81**], Move 7). Since all intersections of SNC^+ divisors are positive, edge signs do not appear and so any moves involving edge signs can be discarded as well ([**Neu81**], Moves 3,5, and 7). Finally, all of our contact 3-manifolds are without boundary so we may discard any moves dealing with graphs with non-empty boundary (Moves 6 and 8). Thus, the only move from Neumann's calculus that is applicable and contact-preserving is the -1 blow-up/blow-down.

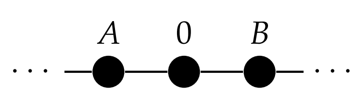
5.2. Contact Plumbing Calculus

The discussion in the previous section tells us that we are disarmed considerably in the contact setting. This makes proving an analog of Lemma 4.5.3 more difficult since it is unclear whether one can reduce divisor graphs to their normal form with -1 blow-ups alone.

Fortunately, since many of Neumann's moves are not-applicable, we merely have to avoid using the $+1$ blow-up/blow-down or $\mathbb{RP}^2$-absorptions to reduce to normal form. The proof of Lemma 4.5.3 ([**Neu81**], Theorem 4.1) crucially uses $+1$ blow-downs to normalize chains and so we must find another way to normalize chains in a way that preserves the contact structure of the divisor boundary. It turns out that -1 blow-ups are enough to get pretty far in the normalization process and draw some conclusions.

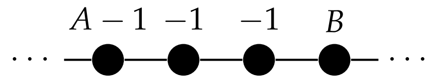
It is convenient to introduce a few additional contact-preserving moves that we can work with. These moves follow directly from applications of -1 blow-ups and blow-downs but it is convenient to give them their own names:

Construction 5.2.1 (0-curve transfer)**.** Consider the following situation

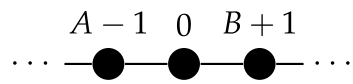


.

We may blow up either edge connected to the 0-curve. For example, we may blow up the left edge

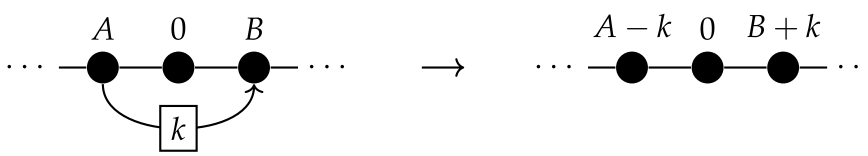


then blow down the (-1)-curve next to the B-curve to arrive at



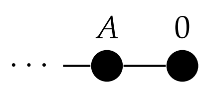
.

Effectively, we have taken from the A-curve and added to the B-curve. This process is called *0-curve transfer*. It's clear that a similar process may be used to transfer from the B-curve to the A-curve. This process can be iterated to increase the transfer between the chains. We will denote transfers with a directed edge

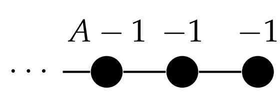


.

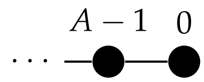
A special case is the following



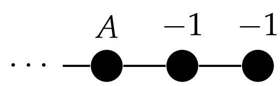
in which we may blow up the internal edge



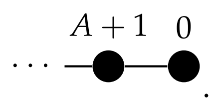
and blow down to



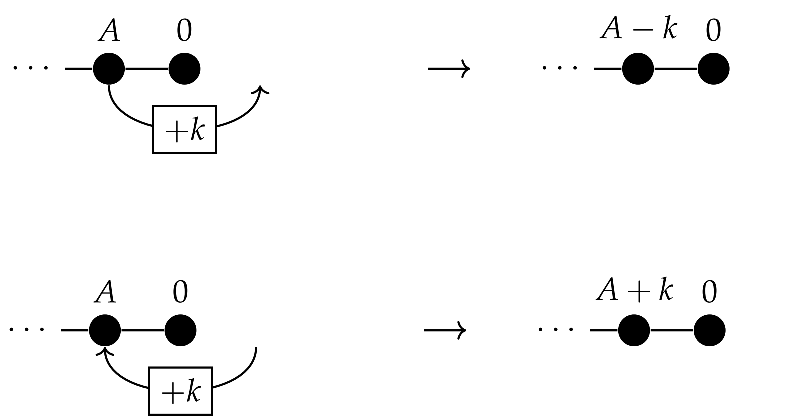
or we may blow up the exterior



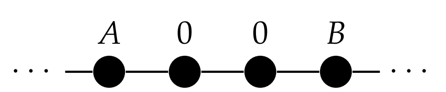
and blow down to



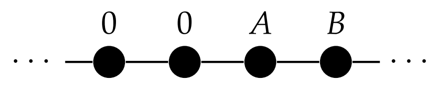
This is referred to as *transferring to/from nowhere* and will be denoted by



Construction 5.2.2 (Sliding)**.** Consider the following situation

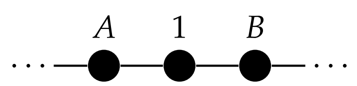


We can use the left 0-curve in the 0-0 configuration to transfer all of the A-curve over. We arrive at

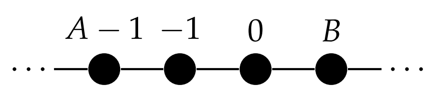


Thus we have moved the 0-0 configuration to the left. This is called *sliding*. It's clear that we can slide in the other direction as well.

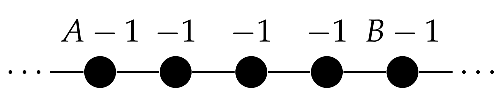
Construction 5.2.3 (Transfer to/from nowhere)**.** We start with



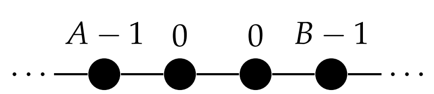
We blow up once on the left to get



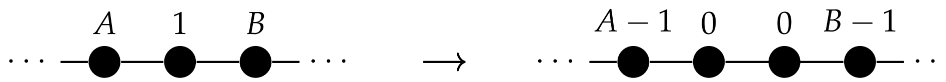
and once on the right to get



We then blow down the middle curve to arrive at



We refer to this process as *chain replacement* and we denote it by



We will use these moves in order to perform a modified chain reduction procedure.

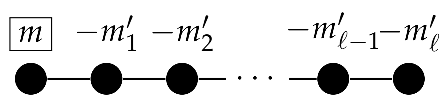
5.3. The Modified Chain Reduction Lemma

We may now prove the following key lemma:

Lemma 5.3.1. *Let* Γ *be a concave divisor graph. Then we may perform a finite sequence of*

- *combinatorial* (-1)*-blow-ups and* (-1)*-blow-downs and*
- *exchanges as in Section 5.2*

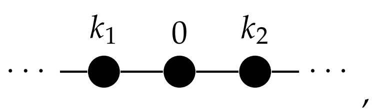
on Γ *to arrive at a graph* Γ^{ξ} *whose maximal chains are all of the form*



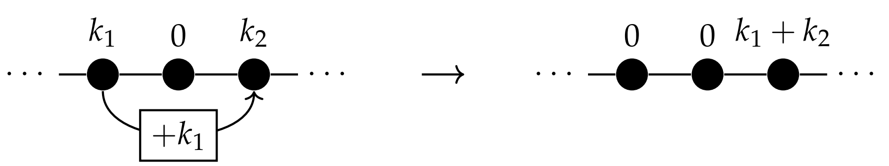
Where the m-vertex represents a linear chain of 0*-curves of length* $m \geq 0$ *and* $m_i \geq 2$*. Such a chain is said to be in* <u>*contact normal form*</u>*. We call* Γ^{ξ} *the* <u>*contact chain reduction of* Γ*.*</u>

PROOF. Let $C \subset \Gamma$ be a maximal chain. Fix some linear ordering on the components of C so that $C = C(m_1, \ldots, m_\ell)$. For exterior maximal chains, we always take m_ℓ to be the decoration on the exterior-most curve. We progress according to the following procedure:

1. Blow down as many -1-curves as possible from left to right (possibly in multiple stages). The result should be a maximal chain with no -1-curves. If the resulting chain is normalized, proceed to the next chain.
2. For each configuration of the form

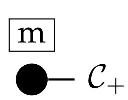


transfer all of k_1 over the 0-curve to the right

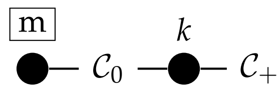


Now shift the 0-0 configuration all the way to the left.

3. Repeat steps (1) and (2), until the chain has no (-1)-curves and no 0-curves except those on the left. Since both steps either preserve or reduce the number of vertices in the chain, this process will eventually halt.
4. The chain is now of the form:

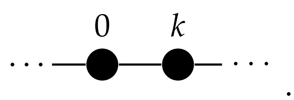


where the $\boxed{\text{m}}$-vertex is a linear chain 0-curves of length m and $\mathcal{C}_+$ is a sub-chain without -1-curves or 0-curves. Starting from the left, identify the first k-curve in the chain with $k > 0$. The chain thus may be decomposed as



where $\mathcal{C}_0$ is a chain with all decorations ≤ -2 and $\mathcal{C}_+$ is the rest of the chain.

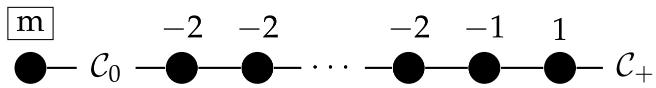
We could have $\mathcal{C}_0 = \emptyset$ and be in the situation:



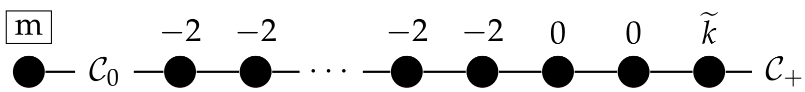
If the number of zeros on the left is odd, we simply use the chain of 0-curves to the left to transfer k into the rest of the graph. This increases the number of 0-curves on the left by one.

If the number of 0-curves on the left is even and non-zero, we may transfer $+2$ from the 0-curve left of the k-curve into the graph using the odd number of 0-curves to its left. This reduces the scenario to the case $\mathcal{C}_0 \neq \emptyset$ and decreases the number of 0-curves on the left by one.

In all other cases, we blow up the k-curve on the left $(k-1)$ times so that it becomes a zero curve, the chain is now of the form:



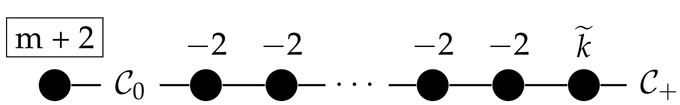
and this process has added $(k-1)$ vertices. We then perform a chain replacement and replace the $+1$-curve with a 0-0 configuration which adds an additional vertex. If $\mathcal{C}_+ \neq \emptyset$, then this leaves the chain in the form



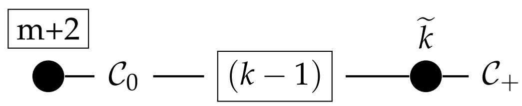
where we have

$$\widetilde{k} = \begin{cases} \leq -2 & \\ -1 & \\ 0 & \\ \geq 1 & \end{cases}.$$

We may slide the 0-0 configuration so that it is of the form

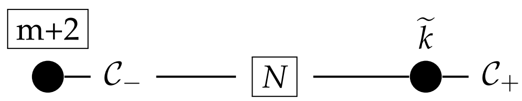


If $\mathcal{C}_+ = \emptyset$, then we proceed to the next chain after sliding. It will be important to notate the chain of -2's for the next step. We will represent our chain by



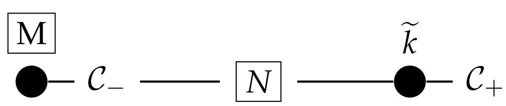
.

By considering the -2-curves at the tail end of $\mathcal{C}_0$, we may further write



where N is a maximal subchain of -2-curves and C_- is a subchain with only negative vertices and no -1-curves.

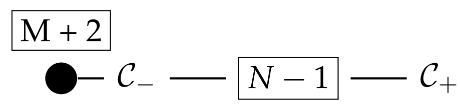
5. If the algorithm did not terminate at the previous step, then we are in the following situation:



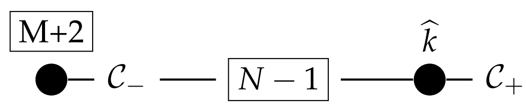
We consider all cases for the value of $\widetilde{k}$ separately:

- $\widetilde{k} \leq -2$ or $\widetilde{k} \geq 1$: For these, we continue to step through the chain or repeat the process outlined in step 4 if necessary.

- $\widetilde{k} = 0$: We proceed as in step 2. At the end of this process, the chain is of the form:

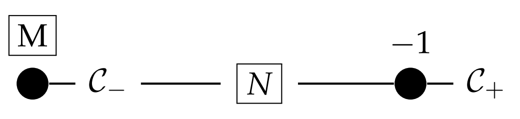


where $\mathcal{C}_+$ is the remainder of the chain. If $\mathcal{C}_+$ is not empty, then the situation is like

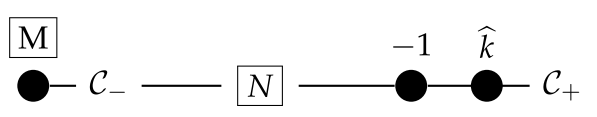


so we may repeat this step for the new value $\widehat{k}$.

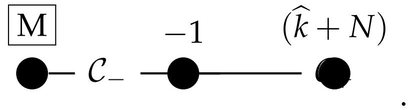
- $\widetilde{k} = -1$: We are in the situation



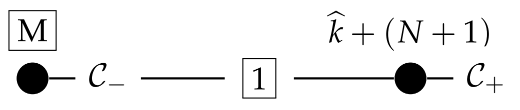
If $\mathcal{C}_+ \neq \emptyset$, then we're actually in the situation



In this case (and even in the case when $\mathcal{C}_+ = \emptyset$) we blow down the -1-curve. Because of the chain of -2's to the left of the -1-curve, we may continue to blow down until the chain is of the form

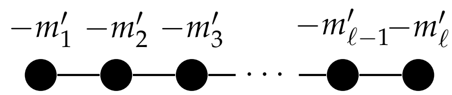


Blowing down the remaining -1 curve will affect the rightmost vertex in the subchain $\mathcal{C}_-$. At worst, this vertex is a -3-curve so that blowing down leave a trailing -2-curve. So, at worst, the situation is

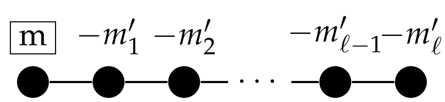


We continue to step through the chain, starting with the $\widehat{k}+(N+1)$ curve.

We continue to repeat the present step until the maximal chain is in contact normal form.



or



where the m-vertex is a linear chain of m 0-curves and $m_i \geq 2$.

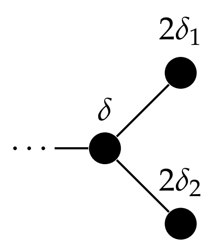
We then repeat the entire procedure for all other chains until all chains are in contact normal form. □

5.4. Contact normal form plumbing graphs

The modified chain reduction lemma shows that we can almost fully normalize a chain in the contact setting using only blow-ups and blow-downs. The only difference between a fully normalized chain and a chain in contact normal form is the admissibility of any number of leading 0-curves. This difference, as well as the issues involving Klein bottle pieces discussed earlier, are the only obstructions to carrying out Neumann's reduction procedure in full. We make the following definition:

Definition 5.4.1. We say that a divisor graph Γ is in *contact normal form* if it is in TPC normal form except that:

- Γ may contain configurations of the form



  where $\delta_i = \pm 1$, $\delta = \frac{\delta_1 + \delta_2}{2}$.
- A chain $C \subset \Gamma$ may contain any number of leading 0-curves.

With this definition and the discussion in the previous section, we have shown

Proposition 5.4.2. *The contact chain reduction* Γ^ξ *of a plumbing graph* Γ *is in contact normal form.*

We refer to Γ^ξ as the *contact reduction* of the graph Γ.

Corollary 5.4.3. *Any plumbing graph* Γ *can be reduced to a graph* Γ^ξ *in contact normal form.*

The two distinguising features of a plumbing graph in contact normal form and a plumbing graph in topological normal form are:

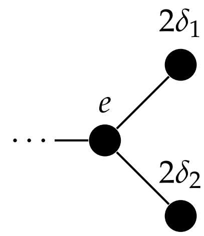
(i) The plumbing graph may have Klein bottle pieces.

(ii) Any chains may contain any number of leading 0-curves.

The first of these conditions has topological implications and the second condition has symplectic and contact topological implications. In the next section, we will analyze the topological ramifications of having an embedded Klein bottle in a divisor boundary. The existence of leading zero chains is closely related to the concept of Giroux torsion from contact geometry. We hope to explicate the details of this in later work. For our applications, it suffices for us to understand the second condition in the context of our particular example (see Section 6.2). We start with the first condition.

5.5. Klein bottle pieces

We turn to the case of plumbing graphs which contain sub-graphs of the form



with $\delta_i = \pm 1$. If we cut Y_Γ along the plumbing torus corresponding to the leftmost edge of this sub-graph, we split Y_Γ into two pieces:

- A piece Y_Γ^* corresponding to the complement of the Klein bottle piece (i.e. the rest of the graph)
- A piece N_K which is diffeomorphic to an orientable tubular neighborhood of a Klein bottle $K \subset N_K \subset Y_\Gamma$.

To see that N_K indeed has this topology, we observe from the corresponding portion of the plumbing graph that N_K is a Seifert-fibered space over a disk with two singular fibers with rational surgery coefficients $\pm\frac{1}{2}$. This gives a Seifert-fibered presentation of the twisted I-bundle over the Klein bottle (which itself is the twisted S^1-bundle over S^1). (cf. **[Hat]**, Chapter 2.1).

The boundary ∂N_K is diffeomorphic to a torus. We say that K is *virtually compressible* if ∂N_K is compressible, otherwise we say that K is *virtually incompressible* or *virtually essential*. For prime 3-manifolds, admitting an embedded virtually compressible Klein bottle puts severe restrictions on their topology.

Proposition 5.5.1. *Suppose that Y is a prime 3-manifold which contains an embedded Klein bottle. If this Klein bottle is virtually compressible, then Y may be given the structure of a Seifert-fibered space over $\mathbb{RP}^2$ with at most one singular fiber.*

PROOF. As above, every Klein bottle piece yields an associated 3-manifold N_K diffeomorphic to an orientable tubular neighborhood of an embedded Klein bottle with ∂N_K diffeomorphic to a torus.

Let $\Delta \subset Y - N_K$ be a compressing disk for ∂N_K. The union $\partial N_K \cup \Delta$ is a torus with a meridional disk attached. It follows that the boundary of a smoothing Y_K of the union of N_k and a neighborhood of Δ is diffeomorphic to a sphere. This allows us to decompose Y_Γ as a connected sum with $Y_K \cup B^3$ appearing as a (non-trivial) summand. Since Y_Γ was assumed to be prime, we know that the complement $Y - Y_K$ is diffeomorphic to a 3-ball.

It follows that $Y - N_K$ is diffeomorphic to a solid torus and so Y is diffeomorphic to a solid torus glued to an orientable tubular neighborhood of a Klein bottle along some diffeomorphism of their torus boundaries. Since N_K has the structure of an S^1-bundle over the Möbius strip, it follows that Y may be given the structure of a Seifert fibration over $\mathbb{RP}^2$ with at most one singular fiber (c.f. **[Hat]**, Theorem 2.3(d)). □

Corollary 5.5.2. *If Y_Γ is a prime plumbing boundary which is not Seifert-fibered, then every Klein bottle piece in Y_Γ contains a virtually essential Klein bottle.*

Thus if a plumbing boundary is not Seifert-fibered, it cannot contain a virtually compressible Klein bottle. While this does not fully deal with the issue of Klein bottle pieces, it does give rise to a simple test for eliminating the possibility of some embedded Klein bottles. To fully manage Klein bottle pieces in the contact setting, one would need to understand the contact-topological implications of the existence of virtually incompressible Klein bottles. We hope to pursue this in future work. For our present purposes, we will avoid analyzing this issue entirely by showing that the divisor boundary in our example cannot admit Klein bottles of any kind, virtually compressible or otherwise. This will be expanded upon in Chapter 6.

CHAPTER 6

Main Results and Applications

In this section, we will collect a number of topological implications for divisor boundaries, compactifications, and neighborhoods that follow from the results in the previous sections. These results are collected with a goal of focusing on the nature of the structure of all divisor compactifications of a given Liouville 4-manifold. We hope that these results will help contribute to the theory of *birational symplectic geometry* in dimension 4.

As an application of these results, we will prove Theorem 1.0.1 which says that the submanifold $X_{KT} \subset M_{KT}$ is not affine. We will discuss how our arguments may be extended to produce other examples and conclude after briefly mentioning further directions for studying non-affine symplectic manifolds.

6.1. Structural theorems and topological implications

Let $(X, d\lambda)$ be a Liouville 4-manifold with finite topology. We will consider the collection of concave SNC^+ divisor compactifications of X. Recall that such a compactification is a pair (M, D) with (M, ω) a closed symplectic manifold and $D \subset M$ a concave SNC^+ divisor such that $M - D$ is symplectic deformation equivalent to X. Each such compactification has an associated contact divisor boundary (Y_D, ξ_D) obtained as the contact-type boundary of a concave neighborhood (N_D, ω) of D. Recall that we say that a divisor D is obstructed if $\Gamma_D^{\xi} \not\approx \Gamma_D^{Top}$ and unobstructed otherwise. Our results allow us to characterize the compactifications of obstructed divisors whose associated divisor boundary Y_D is a prime 3-manifold. If this is the case, then we say that D is *prime*.

Theorem 1.0.3. *Let (M, D) be a concave SNC^+ divisor compactification of X. Then if D is obstructed and prime, we have one or more of the following:*

(i) M is a blow-up of a rational or ruled symplectic manifold,

(ii) the contact 3-manifold $(\partial_\infty X, \xi)$ is Seifert-fibered over $\mathbb{RP}^2$ via a fibration with at most one singular fiber, or

(iii) the contact 3-manifold $(\partial_\infty X, \xi)$ contains an embedded incompressible Klein bottle.

This characterization follows directly from a characterization of concave neighborhoods of obstructed prime divisors.

Theorem 1.0.4. *Let (N_D, ω) be a concave neighborhood of $D \subset M$. If the divisor D is obstructed and prime, then one or more of the following are true:*

(i) up to blow-ups (N_D, ω) is symplectomorphic to a concave divisor neighborhood $(N_{\widetilde{D}}, \widetilde{\omega})$ whose associated divisor contains a 0-curve S such that $S \cdot S = 0$,

(ii) the contact boundary $(\partial N_D, \xi_D)$ is Seifert-fibered over the $\mathbb{RP}^2$ via a fibration with at most one singular fiber, or

(iii) the contact boundary $(\partial N_D, \xi_D)$ *contains an embedded incompressible Klein bottle.*

Proof. Since D is obstructed, $\Gamma_D^{\xi} \neq \Gamma_D^{Top}$. For this to occur, we must have one or both of the following:

- The contact reduction procedure for Γ_D produces chains with leading 0-curves,
- The contact reduction Γ_D^{ξ} contains Klein bottle pieces.

By following the contact reduction procedure for Γ_D, we may perform a sequence of symplectic blow-ups on (N_D, ω) to obtain a concave divisor neighborhood $(N_{\widetilde{D}}, \widetilde{D})$ associated to a concave divisor $\widetilde{D}$ whose associated graph $\Gamma_{\widetilde{D}}$ is isomorphic to the contact reduction Γ_D^{ξ}.

If the first case occurs, then we may conclude that $N_{\widetilde{D}}$ contains a symplectic sphere with self intersection number 0 and so we may conclude (i). If the second case occurs, since D is prime, conclusions (ii) and (iii) follow from Proposition 5.5.1 and Corollary 5.5.2. □

Proof of 1.0.3. This result follows after applying a theorem of McDuff for ruled symplectic manifolds [**McD90**, Corollary 1.5(ii)]. In particular, Theorem 1.0.3(i) follows from the fact that a compact symplectic manifold with a symplectic sphere of self-intersection number 0 is the blow-up of a ruled surface (topologically an S^2 bundle over a closed surface). □

Remark 6.1.1. If one removes the word "prime" from each of the above theorems, then conclusions (ii) and (iii) become less refined and we may only conclude that ∂N_D (or equivalently, the ideal contact boundary $\partial_\infty X$) contains an embedded Klein bottle.

In the unobstructed case, the diffeomorphism type of the contact boundary is sufficient to characterize all concave divisor neighborhoods via direct application of [**Neu81**].

Theorem 1.0.7. Let N_D and $N_{\widetilde{D}}$ denote concave divisor neighborhoods associated to a pair of unobstructed divisors D and $\widetilde{D}$. Then if their divisor boundaries ∂N_D and $\partial N_{\widetilde{D}}$ are diffeomorphic, the manifolds N_D and $N_{\widetilde{D}}$ are diffeomorphic up to blow-ups. □

The combined results about obstructed prime divisors and unobstructed divisors give a complete topological classification of neighborhoods of concave SNC^+ divisors.

With additional consideration, we may apply this to produce our most general result for divisor compactifications. As discussed, the divisor compactification (M, D) admits a decomposition of the form

$$M = \overline{X} \cup_\Phi N_D$$

where $(\overline{X}, d\lambda)$ is a Liouville domain, (N_D, ω) is a concave neighborhood of D and $\Phi\colon (\partial\overline{X}, \xi) \to (\partial N_D, \xi)$ is a contactomorphism defining the gluing. By considering divisor boundaries and ideal contact boundaries, we may consider this to be a contactomorphism $\Phi\colon (\partial_\infty X, \xi) \to (Y_D, \xi_D)$. Thus we may consider Φ to be an element of Cont($\partial_\infty X, \xi$), the contactomorphism group of the ideal contact boundary. The isotopy class $[\Phi] \in \pi_0(\mathrm{Diff}(\partial_\infty X)$ determines the diffeomorphism type of M. This plus Theorem 1.0.7 is enough to conclude:

Theorem 1.0.8. *Let D be a concave compactifying SNC^+ divisor for a 4-dimensional Liouville domain (X^4, ω) and let (M^4, ω) be a compactification of X by an SNC^+ divisor D with associated mapping class $[\Psi] \in \pi_0(Cont(\partial X))$ defining the capping. Then M either satisfies at least one of:*

(i) (M, ω) is a blow-up of a ruled symplectic manifold,

(ii) ∂X is Seifert-fibered over the $\mathbb{RP}^2$ via a fibration with at most one singular fiber,

(iii) ∂X contains a virtually essential Klein bottle

or any other SNC^+ divisor compactification $(\widetilde{M}, \widetilde{\omega})$ with the same mapping class $[\Psi]$ can be obtained from (M, ω) via blow-ups and blow-downs.

This result could be strengthened if one could understand to what extent the contact mapping class $[\Psi]$ may vary. For example, if every diffeomorphism $\Psi\colon \partial N_D \to \partial N_D$ extends to a diffeomorphism $\widetilde{\Psi}\colon N_D \to N_D$, then the diffeomorphism type of any divisor compactification of X is completely determined by the diffeomorphism type of X and the topology of D, independent of the choice of contactomorphism defining the gluing. We will use this to finish a proof of Theorem 1.0.1 in the next section.

6.2. A non-affine Liouville manifold

Let M_{KT} denote the Kodaira-Thurston manifold (see **[Thu76]**) defined as $M_{KT} := \mathcal{M}_\phi \times S^1$ where $\mathcal{M}_\phi$ is the mapping torus of a right Dehn twist $\phi\colon T^2 \to T^2$. We give $\mathcal{M}_\phi$ local coordinates $(\theta_1, \theta_2, \theta_3)$ and so M_{KT} may be given local coordinates $(\theta_1, \theta_2, \theta_3, \theta_4)$ and we define a symplectic form

$$\omega := d\theta_1 \wedge d\theta_2 + d\theta_3 \wedge d\theta_4.$$

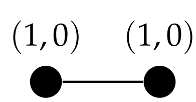
We note that (M_{KT}, ω) contains a natural SNC^+ divisor. Let $\sigma\colon S^1 \to \mathcal{M}_\phi$ be a section of the mapping torus and let $\pi\colon \mathcal{M}_\phi \to S^1$ be the natural projection map. We define two tori

$$D_L := \pi^{-1}(pt) \times \{pt\}$$

$$D_R := \sigma(S^1) \times S^1$$

and let $D := D_L \cup D_R$. Evidently, D_L and D_R are ω-orthogonal symplectic divisors and so D is an SNC^+ divisor.

The plumbing graph associated to the divisor D is



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and so the intersection form is given by

$$Q_D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which admits many GS-positive solutions (regardless of the symplectic area chosen for D_L and D_R). It follows that N_D is concave and so it has contact-type boundary.

We let $X := M_{KT} - D$. Since X is a symplectic punctured torus bundle over a symplectic punctured torus, X is a Liouville manifold [**BHM15**, Theorem 1.2]. The ideal contact boundary $\partial_\infty X$ is contactomorphic to ∂N_D.

Theorem 1.0.1. *The Liouville manifold X is not symplectomorphic to any affine variety.*

Whose proof will rely on the main application of our result.

Proposition 6.2.1. *Any concave SNC^+ compactifying divisor $\widetilde{D}$ for X_{KT} is unobstructed and prime.*

Recall that a divisor $\widetilde{D}$ is unobstructed if $\Gamma^{\xi}_{\widetilde{D}} \approx \Gamma^{Top}_{\widetilde{D}}$ and "prime" simply means that each divisor boundary is a prime 3-manifold. The fact that all compactifying divisors for X are unobstructed allows us to apply Theorem 1.0.7. From there, we verify:

Proposition 6.2.2. *If $\widetilde{D}$ is a compactifying divisor for X_{KT}, then Every diffeomorphism $\Psi\colon \partial N_{\widetilde{D}} \to \partial N_{\widetilde{D}}$ extends to a diffeomorphism $\widetilde{\Psi}\colon N_{\widetilde{D}} \to N_{\widetilde{D}}$.*

By our previous discussion, may conclude:

Theorem 6.2.3. *Let $(\widetilde{M}, \widetilde{\omega})$ be a concave divisor compactification of X_{KT}. Then, up to blow-ups, M is diffeomorphic to M_{KT}.*

From these results, we may prove our main theorem:

PROOF OF THEOREM 1.0.1. Suppose, on the contrary, that X is affine. Then it follows that it admits an algebraic SNC^+ compactification $(\widetilde{M}, \widetilde{D})$. By Proposition 6.2.1 we see that the divisors D and $\widetilde{D}$ are both unobstructed. We may thus apply Theorem 1.0.7 and see that we may perform a sequence of blow-ups on $\widetilde{M}$ until our divisor $\widetilde{D}$ becomes diffeomorphic to D. This process results in a 4-manifold $\widehat{M}$. It then follows from Proposition 6.2.2 that $\widehat{M}$ is diffeomorphic to M_{KT}. This cannot possibly occur since $b_1(M_{KT}) = 3$ and blowing up does not alter the first Betti number. □

We will need a few different lemmas in order to establish Theorem 6.2.3:

(i) The divisor boundary Y_D is a prime 3-manifold

(ii) Y_D does not contain any embedded Klein bottles

(iii) X does not admit a compactification (M, D) such that D contains a sphere with self intersection number 0.

Together, items (i) and (ii) tell us that no divisor with divisor boundary diffeomorphic to Y_D can contain Klein bottle pieces. Item (iii) allow us to avoid issues involving 0-curves as they cannot be a component in any divisor compactification of X. In particular, this implies that no 0-curves appear in the contact chain reduction of any concave divisor graph with divisor boundary diffeomorphic to Y_D. From all of this, it will follow that the contact chain reduction of any such divisor graph must be isomorphic to its topological reduction which is the key to concluding our result. We will prove the lemmas listed above in sequence.

Lemma 6.2.4. *The manifold Y_D is prime.*

Proof. Suppose not and suppose we may compose Y_D as a non-trivial connected sum. Let $S \subset Y_D$ denote the embedded S^2 defining the connected sum decomposition. Recall that Y_D is the result of a gluing of two S^1-bundles over punctured tori. We will let Y_L and Y_R denote these bundles. Let $T \subset Y_D$ be the plumbing torus, i.e. the image of the boundary tori of Y_L and Y_R under this gluing.

We wish to understand how S intersects T. After perturbing via an isotopy, we may assume that S and T intersect transversely. By compactness, we know that S and T intersect along a collection of closed curves. Removing T splits S into a collection of surfaces each having some number of boundary components, each of which is a curve in the collection. Since S is a sphere, one of these pieces must be a disk.

The plumbing torus T is an incompressible surface and so the boundary of this disk must also bound an embedded disk in T. This implies that each of the curves in our collection are contractible in T. Since Y_D is prime, these two disks bound a 3-ball and so we may use this 3-ball to push the disk across T and isotope it until it lies on only one side of the decomposition. We may repeat this argument for the remaining collection and eliminate every intersection, pushing everything to the same side of T. It follows that we may assume that S lies on one side of the plumbing torus or the other. This would allow us to decompose one of the sides as a non-trivial connected sum. Since both $\pi_2(Y_L) \approx \pi_2(Y_R) \approx 0$, we know this is impossible. □

We will now prove that Y_D is free of Klein bottles.

Lemma 6.2.5. *The manifold Y_D contains no embedded Klein bottles.*

PROOF. We start by showing that Y_D cannot contain compressible Klein bottles. By Lemma 6.2.4 and Proposition 5.5.1, if Y_D did contain a virtually compressible Klein bottle, then it may be given the structure of a Seifert fibration with at most one singular fiber. If this were the case, its fundamental group would admit the following presentation:

$$\pi_1(Y_D) = \langle \mu, \lambda : \mu\lambda\mu^{-1} = \lambda^{-1}, \mu^{2a} = \lambda^{-b} \rangle$$

for some $a, b \in \mathbb{Z}$.

We understand the topology of Y_D as the result of plumbing surgery on a pair of trivial S^1-bundles over T^2. From this description, we may give a presentation of the fundamental group (c.f. [**Orl72**, Chapter 5.3]):

$$\pi_1(Y_D) = \langle u, v, x, y, F_L, F_R : F_L = [x, y], F_R = [u, v] \rangle.$$

By abelianizing, we see that

$$H_1(Y_D) = \pi_1(Y_D)_{Ab} = \mathbb{Z}^4$$

so the abelianization of the second presentation is obviously not equal to the abelianization of the first. Thus we arrive at a contradiction.

We then turn to the case of embedded incompressible Klein bottles. Topologically, Y_D is the boundary of a plumbing of two trivial $\mathbb{D}^2$-bundles over T^2. We let $T \subset Y_D$ be a fixed representative of the sole plumbing torus. Suppose that $K \subset Y_D$ is an embedded virtually incompressible Klein bottle.

Cutting Y_D along T splits the manifold into two pieces Y_L and Y_R. Both of these manifolds are diffeomorphic to S^1-bundles over a torus minus an open disk. This cut splits K into a collection $\{\Sigma_i\}_{i \in \mathcal{I}}$ of surfaces with boundary. A fixed boundary component of any one of these surfaces is some closed curve in T. Since K is an essential surface, each Σ_i is an incompressible and boundary incompressible surface in either Y_L or Y_R. These surfaces must be either horizontal or vertical with respect to the Seifert fibrations (c.f. **[Hat]**, Proposition 1.12). Recall a surface Σ in a Seifert-fibered space Y with Seifert-fibration $\pi : Y \to S$ is *vertical* if Σ is a union of fibers of π and *horizontal* if it is transverse to the fibers of π. If Σ is horizontal, the restriction of π gives Σ the structure of a branched cover over S.

Since the plumbing torus T is incompressible, we can assume that none of the Σ_i are disks. Thus since K is broken into the pieces Σ_i by T, we know that each Σ_i must be either:

- a mobius band,
- a punctured Klein bottle,
- an annulus, or
- a Klein bottle.

If Σ_i is a horizontal surface, then it must be a branched cover over a punctured T^2 and hence it must be oriented and have negative Euler characteristic (by the Riemann-Hurwitz formula, [**Jos97**, Theorem 2.5.2]). Thus none of the Σ_i are horizontal.

A vertical surface must be either an annulus or a torus so K is a union of vertical annuli in Y_L and Y_R. In order for these pieces to glue to form a Klein bottle, we must always have a pair of pieces $\Sigma_L \subset Y_L$ and $\Sigma_R \subset Y_R$ which are glued together in the union. Since the gluing between Y_L and Y_R sends the fiber of one side to a section on the other side, we cannot have that both of these surfaces are vertical, a contradiction. □

In order to establish (iii) we need the following.

Lemma 6.2.6. *The cohomology group $H_2(X)$ contains a class with positive self intersection.*

Proof. We consider our chosen section $\sigma\colon S^1 \to \mathcal{M}_\phi$ of the mapping torus and we let $\pi\colon \mathcal{M}_\phi \to S^1$ be the natural projection. As above, we have

$$D_L := \pi^{-1}(pt) \times \{pt\}$$

and

$$D_R := \sigma(S^1) \times S^1.$$

We may trivialize the T^2-bundle $\pi\colon M_{KT} \to T^2$ near the fiber D_L in order to obtain generators for nearby T^2-fibers. Given $q \in S^1$ near our chosen point $p := \pi(D_L)$, we let D_q denote the fiber lying over it. We let $\mu_q, \lambda_q \in H_1(D_q)$ denote the generators obtained from the trivialization. We choose our trivialization such that μ_q is parallel to the curve along which we perform a Dehn twist in each mapping torus $\mathcal{M}_\phi \times \{pt\} \subset M_{KT}$.

We let $A_L \in H_2(M_{KT})$ denote the homology class represented by a product of λ_q with the circle $\{pt\} \times S^1 \subset M_{KT}$. We let $A_R \in H_2(M_{KT})$ denote the homology class represented by a product of a parallel copy of $\sigma(S^1) \times \{pt\}$ (i.e. one disjoint from D_R) with μ, the image of $\mu_q \times I$ in $M_\phi \subset M_{KT}$. We let $A = A_L + A_R$. Evidently, we have $A \cdot A > 0$ which completes the proof. □

With this, we may prove:

Lemma 6.2.7. *X cannot admit a divisor compactification (M, D) such that D contains a sphere of self intersection number zero.*

Proof. Assume by way of contradiction that X admits such a compactification (M, D). Let $S \subset D$ denote a sphere with $S \cdot S = 0$. By [**McD90**, Corollary 1.3(ii)], the manifold M is diffeomorphic to an S^2-bundle over a closed surface Σ blown up some number of times and S is homologous to a fiber of this bundle. It is straightforward to compute the intersection form of this manifold:

Let $\widetilde{M}$ be the minimal symplectic manifold obtained by blowing down all of the exceptional curves of M (so that $\widetilde{M}$ is a genuine S^2-bundle over a surface). The second homology group $H_2(\widetilde{M};\mathbb{Z})$ is generated by a generic fiber F and a generic section $\widetilde{\Sigma} \subset \widetilde{M}$. If $\widetilde{M}$ is trivial, then the intersection form is given in the $\{[F],[\widetilde{\Sigma}]\}$-basis by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and if $\widetilde{M}$ is non-trivial, the intersection form is given by

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

In order to obtain M from $\widetilde{M}$, we must perform blow-ups at a collection of disjoint points $\{p_1,\ldots,p_m\} \subset \widetilde{M}$. We may assume these points are disjoint from F and $\widetilde{\Sigma}$. After performing these blow-ups, the intersection form of M is given by either

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus -\mathbb{1}[m]$$

or

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \oplus -\mathbb{1}[m]$$

where $-\mathbb{1}[m]$ denotes an $m \times m$ matrix with -1's along the diagonal and zeros off the diagonal.

We consider $Z = \widetilde{M} - F$. Since F is parallel to the 0-framed sphere in D, we must have $X \subset Z$. We note that the intersection product

$$- \cdot -\colon H_2(Z) \otimes H_2(Z) \to \mathbb{Z}$$

is negative semi-definite. To see this, we simply observe that Z is diffeomorphic (via our generic section $\widetilde{\Sigma}$) to a blow-up of $\widetilde{\Sigma}^* \times S^2$ where $\widetilde{\Sigma}^* = \widetilde{\Sigma} - \{pt\}$. Every class $\beta \in H_2(Z)$ has $\beta \cdot \beta \leq 0$ so the induced intersection product $H_2(Z) \otimes H_2(Z) \to \mathbb{Z}$ is negative semi-definite. The pushforward of the class from Lemma 6.2.6 along the inclusion is a class with positive self intersection and so we have a contradiction.

□

In order to apply the full strength of our main theorem as in the conclusion of the previous section, we prove:

Proposition 6.2.2. *Every diffeomorphism of Y_D extends to the neighborhood N_D i.e. for any diffeomorphism $\psi\colon Y_D \to Y_D$, there exists a diffeomorphism of $\Psi\colon N_D \to N_D$ such that $\Psi|_{Y_D} = \psi$.*

PROOF. We note that Y_D is not a Seifert fibered space. This follows, for example, from a corollary of Neumann's theorem ([**Neu81**], Corollary 5.7) which states that the normal form of a plumbing graph associated to a Seifert fibered space must be star shaped (i.e. a single genus g component connected to a number of chains of spheres). Γ_D is in normal form but it is certainly not star shaped.

If $T \subset Y_D$ is a plumbing torus representative, then $Y_D - T$ is a pair of S^1-bundles over surfaces with boundary as above. It follows that each piece in this decomposition is Seifert-fibered.

It follows that the singleton set $\{T\}$ yields a JSJ-decomposition of Y_D (c.f. **[Hat]**, Chapter 2). By the uniqueness theorem for JSJ-decompositions, we may assume that ψ is isotopic to a diffeomorphism which fixes a regular neighborhood of T. Thus ψ is isotopic to a diffeomorphism of the two pieces Y_L and Y_R which is the identity near the boundary. This diffeomorphism respects the Seifert fibrations of the two pieces and so is induced by a pair of diffeomorphisms $f_L, f_R \colon T^2_* \to T^2_*$ of the T^2_*, the torus minus a disk. We may use these diffeomorphisms to define our diffeomorphism $\Psi \colon N_D \to N_D$ which completes the proof. □

Once we know that every diffeomorphism extends, we may reach our strongest possible conclusion about the collection of all compactifications.

Theorem 6.2.8. *Let $(M, \widetilde{\omega})$ be a concave divisor compactification of X_{KT}. Then, up to blow-ups, M is diffeomorphic to M_{KT}.*

Proof. Given such a compactification $(M, \widetilde{\omega})$ by a divisor $\widetilde{D}$, we let $\Gamma_{\widetilde{D}}$ denote its associated divisor graph. Since M_{KT} and M are both divisor compactifications of the same manifold X_{KT}, it follows that Y_D and $Y_{\widetilde{D}}$ are diffeomorphic. Since X_{KT} is a Liouville manifold, it follows that (Y_D, ξ_D) and $(Y_{\widetilde{D}}, \xi_{\widetilde{D}})$ are also contactomorphic to $\partial_\infty X_{KT}$, and thus to each other.

Since Y_D is unobstructed, the contact reductions Γ_D^{ξ} and $\Gamma_{\widetilde{D}}^{\xi}$ are both isomorphic to their respective topological reductions. By Neumann's theorem (and the modified chain reduction lemma), it follows that we may relate the divisor graph of Γ_D to $\Gamma_{\widetilde{D}}$ via a sequence of blow-ups and blow-downs. From this, we may perform corresponding blow-ups on $(M, \widetilde{\omega})$ in order to transform the divisor $\widetilde{D}$ into D (this follows because Γ_D is already in topological normal form). This presents a blow-up $B\ell(M)$ as a compactification of X_{KT} by the same divisor D and so M and $B\ell(M)$ differ only by the isotopy class of diffeomorphism of Y_D defining these compactifications. By **??**, there is only one such isotopy class. It follows that $B\ell(M)$ and M_{KT} are diffeomorphic □

Thus we may conclude:

Theorem 1.0.1. *The manifold $X = M_{KT} - D$ is not symplectomorphic to any affine variety.*

A similar argument should work for any Liouville manifold whose ideal contact boundary is prime and whose divisors are all unobstructed, so long as it admits a compactification M with $b_1(M)$ odd. In light of [**McL18**] and [**Sei08**], this is the first example of a non-affine symplectic manifold whose obstruction from being affine cannot be detected by the growth rate of symplectic homology. The existence of this example tells us that there should be a deeper obstruction to being affine that assumes the same basic topological setup as above and is not captured via growth rate techniques. The existence and nature of such an obstruction is still currently unknown.

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